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THEORY OF THE THREE-DIMENSIONAL STRESS STATE IN A PLATE WITH A HOLE

J. B. Alblas

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THEORY OF THE THREE-DIMENSIONAL STRESS STATE IN A PLATE WITH A HOLE

DISSERTATION

for obtaining Degree of Doctor of Technical Sciences at the Technical High School in Delft, according to Article 2 of the Royal Decree of September 16, 1927, State Paper No. 310, under authority of Rector Magnificus Dr. O. Bottema, Professor at the Department of General Sciences, Defended Before a Committee of the Senate on July 3, 1957, at 4 p.m., given

by

Johannes Bartholomeus Albias
born in Gravenhage

H. J. Paris - Amsterdam 1957

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This dissertation has been accepted under the supervisor:
Prof. Dr. W. T. Koiter

The research for this dissertation was undertaken using the test facilities at the Laboratory for Applied Mechanics of the Department of Structural Science of the Technical High School.

I would like to express my appreciation to this laboratory, which made it possible for me to publish the results of this research in this form.

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THEORY OF THE THREE-DIMENSIONAL STRESS STATE IN A PLATE WITH A HOLE

J. B. Alblas

For the special cases of stretching and bending of an infinite plate with a circular cylindrical hole, a rigorous analysis is developed in this thesis. The method of investigation is based on the use of complex eigenfunctions, whose general theory was developed some years ago by Green (Ref. 5). It is the purpose of this investigation to examine the effect of the plate-thickness on the stress distribution and to discuss the errors due to the approximate character of current two-dimensional theories.

INTRODUCTION AND SUMMARY /IX*

In the classical theory of elasticity, it is customary to discuss the problems pertaining to plates loaded in, or perpendicular to, their plane on the basis of a two-dimensional theory. For this purpose, the general three-dimensional equations -- for which there is only a small number of particular cases for which an exact solution is known -- are replaced by a system in which the unknowns only depend on the coordinates along the surface of the plate. On the basis of this simplification, certain assumptions are postulated on the basis of physical considerations regarding the dependence of the tension intensity of the coordinate in the normal direction. However, it is a fact that the approximate nature of the two-dimensional solution is not always realized in the literature when discussing problems of stress in plates of arbitrary thickness. For this reason, one of the purposes of this report is to submit the customary computational methods to a critical investigation, and to provide an evaluation of the mistakes which result from the assumptions employed.

In order to arrive at quantitative results, a few special plate problems will be discussed which are related to traction and flexure of cylindrically-perforated isotropic plates. It will be assumed that the influence of the thickness of these plates on the stress distribution is not known. In order to simplify the calculations, it will be assumed that the plates are infinitely extended and stressed along the edges.

The perforated plate, which is loaded in tension, has already been investigated several times. Kirsch (Ref. 1)(1) was the first one to calculate the influence of a small circular hole on the stress distribution in a very thin or a very thick plate. The solution given by Kirsch, which is of important

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* Numbers in the margin indicate pagination in the original foreign text.

(1) Numbers between the parentheses refer to the literature on page 91.

historical significance, describes the stress distribution in the plate as a plane state of deformation. The theory of the general state of plane-stress was given by Silon (Ref. 2) a few years later. (Also see (Ref. 3), § 94 and 146). Reissner (Ref. 9) has given an approximate theory for the deviations existing between the state of stress occurring in reality, and the stresses /X which are found, on the basis of the theory of generalized plane-state of stress. Reissner indicates the points at which the corrections to the classical values may become significant, but he does not provide numerical results.

Green (Ref. 4) was the first author to point out the necessity of testing the accuracy provided by the theory of the generalized plane state of stress as opposed to the exact solution of the three-dimensional problem. For this purpose, he has developed a theory for the plate of finite thickness with circular-cylindrical perforations which is loaded in tension. For the case in which the thickness of the plate equals the diameter of the hole, the formulas have been worked out numerically. As a result of this process, he arrived at two important conclusions -- namely, that the average value of the concentration of tension at the hole, according to the three-dimensional theory, is in good accord with the similar value provided by the generalized theory of plane stress. However, as might be expected, the value of a stress component in the immediate proximity of a border plane deviates rather considerably from the average value over the thickness. One disadvantage inherent to the method advanced by Green, as he developed in (Ref. 4), is the fact that this method results in a system of equations whose coefficients cannot be readily established. As a result of this, the extension to other ratios of the thickness to the hole diameter becomes difficult. For this reason, Green has, in a later publication (Ref. 5), pointed out another method by means of which numerical results can be obtained more readily. This method presents a development toward complex eigen functions for general, three-dimensional plate problems, and it provides the basis of the method of eigen functions to be developed in this report.

The methods which have been developed by Green have one disadvantage in the fact that the convergence of the occurring series becomes worse when the thickness is increased. A method for calculating the three-dimensional states of stress, which may be applied to all thicknesses, has been developed by Sternberg and Sadowsky (Ref. 6). They approached the problem with the aid of stress functions which satisfy the equilibrium conditions and the boundary conditions, and employ the method of Ritz. Their results provide adequate qualitative insight, but they are not always quantitatively correct due to the computational method which is employed.

The first calculations performed for the perforated plate loaded in flexure were performed by Bickley (Ref. 7) and Goodier (Ref. 8). Bickley /XI started with the generalized state of plane stress, and obtained the solution which is called an "elementary" solution in this report. He determines the constants which have to be established by means of the two boundary conditions given by Kirchhoff. The work of Goodier is more general, because the influence of elliptic holes is also discussed. However, it does not surpass the work of Bickley with regard to the circular hole. Reissner (Ref. 10) has developed a more general theory for the flexure of thin plates, with which three boundary conditions can be fulfilled on a boundary. In this theory, which expands upon the report (Ref. 9) of Reissner, a simple dependence on the thickness-coordinate is used for the stresses. In addition, a variational principle is employed for the deduction of the fundamental equations. Reissner has applied his theory to the flexure of a circularly-perforated plate. He finds deviations from the classical theory which are important. In his publication (Ref. 5), Green has clarified the connection between the theory of Reissner and the precise theory. In Chapter III, we shall discuss this matter in greater detail.

When one applies the experimental method for determining the residual stresses advanced by Mathar (Ref. 11) and Soete (Ref. 12, 13, 14), a number of incorrect elements is introduced. One of these is the result produced by employing the formulas of the theory of plane-stress. [See also in this respect, (Ref. 15, 16).] Consequently, on the basis of the three-dimensional theory, it is desirable to establish the influence of the plate thickness upon the stress distribution at the periphery of the hole.

The purpose of the investigation described in this report is to continue the work performed in the references mentioned above. Chapter I disucsses the problem of the circularly-perforated, infinitely extended isotropic plate, which is loaded in tension. The solution of this problem is sought in the form of a superposition of a state of plane stress and an infinite series of three-dimensional stress states. An infinite system of equations is derived for the coefficients which multiply each of these stress states. In the case of the very thin plate, asymptotic developments are given for these coefficients. Since all stress states are derived from potential functions, an appendix has been added to Chapter I, which illustrates the comprehensive nature of the system of potential functions employed.

The results derived from the numerical calculations of the problem presented in Chapter I are discussed in Chapter II. It is apparent that these /XII numerical calculations become continuously more extensive, in proportion to the increase of the plate thickness. The reason for this lies in the fact that in an increasing ratio of thickness to diameter, the number of three-dimensional stress states in the infinite series which has to be included in the calculation increases considerably. On the other hand, it is apparent that in the case of a plate thickness, which is double the diameter of the hole, the state of plane deformation already occurs in the center of the plate. It can be expected that above this ratio the values of the stresses at the boundary planes, as well as in the proximity of the hole, will not change considerably as a result of further increases in the thickness. This has been confirmed by experimental investigations. For this reason, the calculations have been performed up to a thickness which is twice as large as the diameter of the hole. The results can then be extrapolated to very thick plates with reasonable exactitude.

The flexure of a perforated plate is discussed in Chapter III. The theory advanced for this parallels the theory presented in Chapter I, for the most part. In this case, particular attention is also given to the limiting case of the thin plate. The relationship between the exact theory and the theory of Reissner is discussed extensively, as a result of which new insights are obtained with regard to the work of Green (Ref. 5).

The numerical results derived from the calculations performed in Chapter III are presented in Chapter IV. For a thick plate, where the higher terms in the development cannot be neglected, the results are compared with those obtained on the basis of the theory of Reissner. It becomes clear that the exact value of the stress concentration factor at the *boundary plane* is less than that derived from the theory of Reissner. However, this approximate theory provides an excellent prediction of the concentration factor for the tangential flexure *moment*.

In Chapter V the calculations are applied during a theoretical discussion of the method advanced by Mathar-Soete. The influence of the thickness upon the interpretation of the observations is calculated for the tension loading, as well as for flexure. This proves to be small for tension, and large for flexure.

CHAPTER I /1

THEORY OF THE PLATE STRESSED IN ITS PLANE

1.1 Introduction

The treatment of a problem in the three-dimensional theory of elasticity of a homogeneous isotropic body, in which no mass forces are operating, means mathematically that we must try to determine a solution of a system of partial differential equations of the sixth order (Ref. 17)

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{1-2\nu}\,\frac{\partial \varepsilon_v}{\partial x} &= 0, \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{1-2\nu}\,\frac{\partial \varepsilon_v}{\partial y} &= 0, \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + \frac{1}{1-2\nu}\,\frac{\partial \varepsilon_v}{\partial z} &= 0, \end{aligned}\right] \tag{1.1}$$

which, on the surface of the body, has to satisfy three boundary conditions. Here (x, y, z) are the three Cartesian coordinates, (u, v, w) are the displacements in the x-y-z-system, v is the Poisson number, and ε_v is the volume extension, given by

$$\varepsilon_v = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{1.2}$$

A number of exact solutions is known for the system (1.1) (with the boundary conditions): Either some of the displacements have a simple form, or the equations can be reduced to a more simple form as a result of established symmetry. In the case which will be discussed, however, the exact solutions of (1.1) will be obtained only in the form of an infinite series.

It is expedient not to start directly from equations (1.1), but to integrate these equations using potential functions. The potential functions used in this report were introduced for the first time by Boussinesq (Ref. 18) and were later used by Dougall (Ref. 34) with this type of problems. A general discussion of the integration of (1.1) by means of potential functions will be found in the appendix of this chapter. /2

1.2 The Potentials

Three potential functions are used: A, B_1 and B_2, all of which are solutions of the equation of Laplace in which

$$\triangle (A, B_1, B_2) = 0 \tag{1.3}$$

from which we have

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

If the displacements (u, v, w) are expressed by means of the potential function A through the following equations

$$u = \frac{\partial A}{\partial y}, \quad v = -\frac{\partial A}{\partial x}, \quad w = 0, \tag{1.4}$$

equation (1.1) will be satisfied identically, while

$$\varepsilon_v = 0. \tag{1.5}$$

This is likewise the case for displacements (u, v, w), derived from a potential function B_1 through

$$u = \frac{\partial B_1}{\partial x}, \quad v = \frac{\partial B_1}{\partial y}, \quad w = \frac{\partial B_1}{\partial z}. \tag{1.6}$$

Also in this case we have

$$\varepsilon_v = 0. \tag{1.7}$$

A third system of displacements (u, v, w) is derived from a potential function B_2 through

$$\begin{aligned} u &= \alpha \frac{\partial B_2}{\partial x} + z \frac{\partial^2 B_2}{\partial x \partial z}, \\ v &= \alpha \frac{\partial B_2}{\partial y} + z \frac{\partial^2 B_2}{\partial y \partial z}, \\ w &= \gamma \frac{\partial B_2}{\partial z} + z \frac{\partial^2 B_2}{\partial z^2}. \end{aligned} \tag{1.8}$$

In order that these displacements may be solutions of (1.1), a relation must exist between α and γ /3

$$\gamma - \alpha = -3 + 4\nu, \tag{1.9}$$

with which we obtain

$$\varepsilon_v = -2(1-2\nu)\frac{\partial^2 B_2}{\partial z^2}. \tag{1.10}$$

One of the constants α and γ can be freely chosen.

In the following we shall employ

$$\begin{aligned} \alpha &= 2(1-\nu), \\ \gamma &= -(1-2\nu). \end{aligned} \tag{1.11}$$

In some cases, another choice of the constants α and γ is more expedient -- namely,

$$\left.\begin{aligned} \alpha &= 1-2\nu, \\ \gamma &= -2(1-\nu), \\ \text{respectively} \qquad \alpha &= 0, \\ \gamma &= -3+4\nu. \end{aligned}\right| \tag{1.12}$$

The potentials A, B_1 and B_2 are always independent. This may be seen in the fact that with A, the values of ε_v and σ_z are zero everywhere, with B_1 only ε_v is zero everywhere, but not σ_z. With B_2, neither ε_v nor σ_z disappear identically. In the appendix it is proven that the potential function system (A, B_1, B_2) is also complete for the problems to be discussed in this report.

1.3 Transformation to Cylindrical Coordinates

For the problem of the plate with a cylindrical hole loaded in tension, it is expedient to introduce the cylindrical coordinates (r, ϕ, z). The displacements (u_r, u_ϕ, u_z) are derived from the potentials using the following formulae

$$u_r = \frac{1}{r}\frac{\partial A}{\partial \varphi},\quad u_\varphi = -\frac{\partial A}{\partial r},\quad u_z = 0; \tag{1.13}$$

$$u_r = \frac{\partial B_1}{\partial r},\quad u_\varphi = \frac{1}{r}\frac{\partial B_1}{\partial \varphi},\quad u_z = \frac{\partial B_1}{\partial z}; \tag{1.14}$$

$$\left.\begin{aligned} u_r &= \alpha\frac{\partial B_2}{\partial r} + z\frac{\partial^2 B_2}{\partial r\,\partial z}, \\ u_\varphi &= \frac{\alpha}{r}\frac{\partial B_2}{\partial \varphi} + \frac{z}{r}\frac{\partial^2 B_2}{\partial \varphi\,\partial z}, \\ u_z &= \gamma\frac{\partial B_2}{\partial z} + z\frac{\partial^2 B_2}{\partial z^2}. \end{aligned}\right] \tag{1.15}$$

The connection between the stresses and the displacements is expressed by the following known equations (Ref. 3) /4

$$
\left.
\begin{aligned}
\frac{\sigma_r}{2G} &= \frac{\partial u_r}{\partial r} + \frac{\nu}{1-2\nu}\varepsilon_v, \\
\frac{\sigma_\varphi}{2G} &= \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{\nu}{1-2\nu}\varepsilon_v, \\
\frac{\sigma_z}{2G} &= \frac{\partial u_z}{\partial z} + \frac{\nu}{1-2\nu}\varepsilon_v, \\
\frac{\tau_{\varphi z}}{2G} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \varphi} + \frac{\partial u_\varphi}{\partial z}\right), \\
\frac{\tau_{rz}}{2G} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \\
\frac{\tau_{r\varphi}}{2G} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right).
\end{aligned}
\right] \tag{1.16}
$$

In this (σ_r, σ_ϕ, σ_z $\tau_{\phi z}$, τ_{rz} and $\tau_{r\phi}$) are the tensions in cylindrical coordinates, G is the shear modulus, while the volume strain ε_v in cylindrical coordinates is given by

$$\varepsilon_v = \frac{1}{r}\frac{\partial}{\partial r}(r\,u_r) + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z}. \tag{1.17}$$

With the aid of the equations (1.13), (1.14), (1.15), (1.16) and (1.17), we may readily obtain

for the function A /5

$$
\left.
\begin{aligned}
\frac{\sigma_r}{2G} &= \frac{1}{r}\frac{\partial^2 A}{\partial r\,\partial \varphi} - \frac{1}{r^2}\frac{\partial A}{\partial \varphi}, \\
\frac{\sigma_\varphi}{2G} &= -\frac{\sigma_r}{2G} = -\frac{1}{r}\frac{\partial^2 A}{\partial r\,\partial \varphi} + \frac{1}{r^2}\frac{\partial A}{\partial \varphi}, \\
\sigma_z &= 0, \\
\frac{\tau_{z\varphi}}{2G} &= -\frac{1}{2}\frac{\partial^2 A}{\partial r\,\partial z}, \\
\frac{\tau_{rz}}{2G} &= \frac{1}{2r}\frac{\partial^2 A}{\partial \varphi\,\partial z}, \\
\frac{\tau_{r\varphi}}{2G} &= \frac{1}{2r}\frac{\partial A}{\partial r} - \frac{1}{2}\frac{\partial^2 A}{\partial r^2} + \frac{1}{2r^2}\frac{\partial^2 A}{\partial \varphi^2};
\end{aligned}
\right] \tag{1.18}
$$

for the function B_1

$$
\left.
\begin{aligned}
\frac{\sigma_r}{2G} &= \frac{\partial^2 B_1}{\partial r^2}, \\
\frac{\sigma_\varphi}{2G} &= -\frac{\partial^2 B_1}{\partial r^2} - \frac{\partial^2 B_1}{\partial z^2}, \\
\frac{\sigma_z}{2G} &= \frac{\partial^2 B_1}{\partial z^2}, \\
\frac{\tau_{\varphi z}}{2G} &= \frac{1}{r}\frac{\partial^2 B_1}{\partial \varphi \partial z}, \\
\frac{\tau_{rz}}{2G} &= \frac{\partial^2 B_1}{\partial r \partial z}, \\
\frac{\tau_{r\varphi}}{2G} &= \frac{1}{r}\frac{\partial^2 B_1}{\partial r \partial \varphi} - \frac{1}{r^2}\frac{\partial B_1}{\partial \varphi};
\end{aligned}
\right] \tag{1.19}
$$

for the function B_2 /6

$$
\left.
\begin{aligned}
\frac{\sigma_r}{2G} &= 2(1-\nu)\frac{\partial^2 B_2}{\partial r^2} - 2\nu\frac{\partial^2 B_2}{\partial z^2} + z\frac{\partial^3 B_2}{\partial r^2 \partial z}, \\
\frac{\sigma_\varphi}{2G} &= -2(1-\nu)\frac{\partial^2 B_2}{\partial r^2} - 2\frac{\partial^2 B_2}{\partial z^2} - z\frac{\partial^3 B_2}{\partial r^2 \partial z} - z\frac{\partial^3 B_2}{\partial z^3}, \\
\frac{\sigma_z}{2G} &= z\frac{\partial^3 B_2}{\partial z^3}, \\
\frac{\tau_{\varphi z}}{2G} &= \frac{1}{r}\frac{\partial^2 B_2}{\partial z \partial \varphi} + \frac{z}{r}\frac{\partial^3 B_2}{\partial z^2 \partial \varphi}, \\
\frac{\tau_{rz}}{2G} &= \frac{\partial^2 B_2}{\partial r \partial z} + z\frac{\partial^3 B_2}{\partial r \partial z^2}, \\
\frac{\tau_{r\varphi}}{2G} &= \frac{2(1-\nu)}{r}\frac{\partial^2 B_2}{\partial \varphi \partial z} - \frac{2(1-\nu)}{r^2}\frac{\partial B_2}{\partial \varphi} + \\
&\quad + \frac{z}{r}\frac{\partial^3 B_2}{\partial r \partial \varphi \partial z} - \frac{z}{r^2}\frac{\partial^2 B_2}{\partial \varphi \partial z}.
\end{aligned}
\right] \tag{1.20}
$$

1.4 The Tension-Loaded Perforated Plate

After the preparations in the preceding sections, we can now pass on to the actual problem. We shall examine a uniform isotropic plate whose median plane coincides with the x-y plane and which is delimited by the planes $z = \pm h$. The plate extends in both the x- and the y-direction from $-\infty$ to $+\infty$. The plate is pierced by a cylinder of radius a; the cylinder's median line runs along the z-axis. The plate is tension-loaded in the x-direction by infinite, uniformly distributed stress T.

Mathematically, the problem is defined by equations (1.1) with the boundary conditions /7

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \quad \text{for} \quad z = \pm h, \tag{1.21}$$

$$\sigma_x = T, \sigma_y = \tau_{xy} = 0 \quad \text{for } r = \sqrt{x^2 + y^2} \to \infty; \tag{1.22}$$

$$\sigma_r = \tau_{rz} = \tau_{r\varphi} = 0 \quad \text{along hole edge} \quad r = a. \tag{1.23}$$

At a great distance from the hole, a plane state of stress prevails. This is defined in cylindrical coordinates by

$$\left.\begin{aligned} \sigma_r &= 1/2\,T(1 + \cos 2\varphi), \\ \sigma_\varphi &= 1/2\,T(1 - \cos 2\varphi), \\ \tau_{r\varphi} &= -\,1/2\,T \sin 2\varphi, \\ \sigma_z &= \tau_{rz} = \tau_{r\varphi} = 0. \end{aligned}\right] \tag{1.24}$$

In the vicinity of the hole, stress state (1.24) is disturbed. On stress state (1.24), a stress distribution is now superposed which strives to zero at infinity and frees the boundary from stress.

1.5 Elementary Stress Distributions

Use will be made of several simple solutions of expression (1.3). Expression (1.3) is transformed into

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\,\frac{\partial f}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} = 0, \tag{1.25}$$

for cylindrical coordinates; here f is written for an arbitrary potential. One solution of expression (1.25), which is independent of z and ϕ, is

$$f = c_1 \log r + c_2. \tag{1.26}$$

If solution (1.26) is chosen for function B_1, the equations (1.19) then give rise to

$$\left.\begin{aligned} \frac{\sigma_r}{2G} &= -\frac{c_1}{r^2}, \\ \frac{\sigma_\varphi}{2G} &= +\frac{c_1}{r^2}, \\ \sigma_z &= \tau_{\varphi z} = \tau_{rz} = \tau_{r\varphi} = 0. \end{aligned}\right] \tag{1.27}$$

A solution of expression (1.25) which is independent of z, but dependent of cos 2ϕ, is /8

$$f = \frac{c_3}{r^2} \cos 2\varphi. \tag{1.28}$$

If the solution (1.28) is now chosen for function B_1, the equations (1.19) become

$$\left.\begin{aligned}
\frac{\sigma_r}{2G} &= \frac{6c_3}{r^4}\cos 2\varphi, \\
\frac{\sigma_\varphi}{2G} &= -\frac{6c_3}{r^4}\cos 2\varphi, \\
\frac{\tau_{r\varphi}}{2G} &= \frac{6c_3}{r^4}\sin 2\varphi, \\
\sigma_z &= \tau_{zr} = \tau_{z\varphi} = 0.
\end{aligned}\right] \tag{1.29}$$

Other solutions of expression (1.25) are

$$f = c_4\left(\frac{z^2}{r^2} + \frac{1}{2}\right)\begin{cases}\sin 2\varphi \\ \cos 2\varphi\end{cases} \tag{1.30}$$

If now

$$A = 4c_4\left(\frac{z^2}{r^2} + \frac{1}{2}\right)\sin 2\varphi,$$

$$B_2 = c_4\left(\frac{z^2}{r^2} + \frac{1}{2}\right)\cos 2\varphi,$$

is chosen, superposition of stress values in accordance with expressions (1.18) and (1.20) leads to

$$\left.\begin{aligned}
\frac{\sigma_r}{2G} &= -c_4\left[\frac{4(1+\nu)}{r^2} + \frac{12\nu z^2}{r^4}\right]\cos 2\varphi, \\
\frac{\sigma_\varphi}{2G} &= c_4\frac{12\nu z^2}{r^4}\cos 2\varphi, \\
\frac{\tau_{r\varphi}}{2G} &= -c_4\left[\frac{2(1+\nu)}{r^2} + \frac{12\nu z^2}{r^4}\right]\sin 2\varphi, \\
\sigma_z &= \tau_{zr} = \tau_{z\varphi} = 0.
\end{aligned}\right] \tag{1.31}$$

The stress distributions (1.24), (1.27), (1.29) and (1.31) are four /9 plane states of stress. These are superposed, and in so doing we set

$$\left.\begin{aligned}
c_1 &= \frac{Ta^2}{4G}, \\
c_3 &= \frac{Ta^4}{2G}.C, \\
c_4 &= \frac{Ta^2}{2G}.D.
\end{aligned}\right] \tag{1.32}$$

C and D are constants which are to be still more accurately determined. Thus, we obtain

$$\left.\begin{aligned}
\frac{\sigma_r}{T} &= \frac{1}{2}\left(1-\frac{a^2}{r^2}\right)+\cos 2\varphi\left\{\frac{1}{2}+6C\frac{a^4}{r^4}-\right.\\
&\qquad\left.-D\left[4(1+\nu)\frac{a^2}{r^2}+\frac{12\,\nu a^2 z^2}{r^4}\right]\right\},\\
\frac{\sigma_\varphi}{T} &= \frac{1}{2}\left(1+\frac{a^2}{r^2}\right)+\cos 2\varphi\left\{-\frac{1}{2}-6C\frac{a^4}{r^4}+\right.\\
&\qquad\left.+D\frac{12\,\nu a^2 z^2}{r^4}\right\},\\
\frac{\tau_{r\varphi}}{T} &= \sin 2\varphi\left\{-\frac{1}{2}+6C\frac{a^4}{r^4}-D\left[2(1+\nu)\frac{a^2}{r^2}+\right.\right.\\
&\qquad\left.\left.+\frac{12\,\nu a^2 z^2}{r^4}\right]\right\},\\
\sigma_z &= \tau_{z\varphi} = \tau_{zr} = 0.
\end{aligned}\right] \tag{1.33}$$

It is clear that if $|z| \ll a$ -- a condition which is fulfilled for all values of z by $h \ll a$ -- the z^2 terms in expression (1.33) may be neglected. In this case, the solution to this problem is given by

$$\left.\begin{aligned}
C &= \frac{1}{4},\\
D &= \frac{1}{2(1+\nu)}.
\end{aligned}\right] \tag{1.34a}$$

In the generalized plane state of stress, it is required that the average stress values over z satisfy the boundary conditions. We then find that /10

$$\left.\begin{aligned}
C &= \frac{1}{4}+\frac{\nu}{3(1+\nu)}\frac{h^2}{a^2},\\
D &= \frac{1}{2(1+\nu)}.
\end{aligned}\right] \tag{1.34b}$$

The formulas for the generalized plane state of stress are exactly fulfilled if the half plate-thickness h is small in relation to the radius of the hole a. When this condition is not met, nothing more can be accomplished by fulfilling the boundary conditions for the averaged stresses alone. However, the requirement must be stipulated that these boundary conditions at all points -- i.e., for all values of z -- be fulfilled by $|z| < h$. A number of other stress distributions must be added to stress distribution (1.33). The nature of these added stress distributions is not elementary, and the distributions are derived from more general solutions to equation (1.25). It should be noted that by expression (1.33), the stresses each consist of two parts -- one part independent of ϕ, and one proportional to cos 2ϕ or sin 2ϕ. The first part already satisfies the boundary conditions along the hole. The stress distribution to be added is proportional to cos 2_ϕ and sin 2_ϕ.

1.6 The Eigen Functions

From the foregoing it immediately follows that potential functions having the form

$$f(r, \varphi, z) = g(r, z) \begin{matrix}\cos\\ \sin\end{matrix} 2\varphi, \tag{1.35}$$

must be chosen, where g fulfills

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r} \frac{\partial g}{\partial r} - \frac{4}{r^2} g + \frac{\partial^2 g}{\partial z^2} = 0. \tag{1.36}$$

A general class of solutions to expression (1.36) was obtained by separating the variables. These solutions have the form /11

$$g = \begin{Bmatrix} K_2(\lambda r) \\ I_2(\lambda r) \end{Bmatrix} \begin{Bmatrix} \cos \lambda z \\ \sin \lambda z \end{Bmatrix} \tag{1.37}$$

in which $I_2(\lambda r)$ and $K_2(\lambda r)$ are modified second-order Bessel functions (Refs. 19, 20)*, and λ is an arbitrary complex parameter. Of these solutions, only those with the Bessel function of the second kind (K_2-~~function) -- and indeed~~ only those for which the real part of λ is positive -- are taken into consideration when it is stipulated that the corrections to the elementarily computed stress distributions must tend to zero for $r \to \infty$. An attempt has been made to formulate combinations of these solutions such that each of these combinations fulfills the boundary conditions (1.21).

First of all, the following equation is established

$$A = \frac{T a^2}{2 G} K_2(\lambda r) \cos \lambda z \sin 2\varphi . \tag{1.38}$$

By means of expression (1.18) it follows from this that for $|z| = h$

$$\left.\begin{aligned} \sigma_z &= 0, \\ \frac{\tau_{z\varphi}}{T} &= \pm \frac{\lambda^2 a^2}{2} K_2'(\lambda r) \sin \lambda h \sin 2\varphi, \\ \frac{\tau_{rz}}{T} &= \mp \lambda a^2 \frac{K_2(\lambda r)}{r} \sin \lambda h \cos 2\varphi. \end{aligned}\right] \tag{1.39}$$

---

* The definition of $K_n(z)$ used here is the one given in (Ref. 20) on page 373. The definition according to (Ref. 19) deviates from this by a factor of cos $n\pi$ [see also (Ref. 19], page 78).

Boundary conditions (1.21) are fulfilled for every (r, ϕ) if

$$\lambda = \frac{k\pi}{h}. \qquad (k = 1,2,\ldots.) \tag{1.40}$$

A very general solution of expression (1.31) which fulfills boundary condi- /12
tions (1.21) is now obtained by combining these solutions linearly. We thus have

$$A = \frac{T a^2}{2G} \sum_{k=1}^{\infty} a_k \frac{K_2\left(\frac{k\pi r}{h}\right)}{K_2\left(\frac{k\pi a}{h}\right)} \cos\frac{k\pi z}{h} \sin 2\varphi, \tag{1.41}$$

in which we may still freely deal with coefficients a_k. Solutions to the potentials B_1 and B_2 are found in a similar fashion. These must be simultaneously solved. With constants B_1^* and B_2^* it is assumed that

$$\left.\begin{aligned} B_1 &= B_1^* \frac{T a^2}{2G} K_2(\lambda r) \cos\lambda z \cos 2\varphi, \\ B_2 &= B_2^* \frac{T a^2}{2G} K_2(\lambda r) \cos\lambda z \cos 2\varphi; \end{aligned}\right] \tag{1.42}$$

By means of expressions (1.19) and (1.20) the boundary plate stresses for $|z| = h$ are determined. It is found that

$$\left.\begin{aligned} \frac{\sigma_z}{T} &= a^2\cos 2\varphi\, K_2(\lambda r)\, \{-B_1^* \lambda^2 \cos\lambda h + B_2^* \lambda^2 h \sin\lambda h\}, \\ \frac{\tau_{rz}}{T} &= \mp a^2\cos 2\varphi\, K_2'(\lambda r)\, \{B_1^* \lambda^2 \sin\lambda h + B_2^* \lambda^2 \sin\lambda h + \\ &\qquad + B_2^* \lambda^3 h \cos\lambda h\}, \\ \frac{\tau_{\varphi z}}{T} &= \pm 2a^2 \sin 2\varphi\, \frac{K_2(\lambda r)}{r} \{B_1^* \lambda \sin\lambda h + B_2^* \lambda \sin\lambda h + \\ &\qquad + B_2^* \lambda^2 h \cos\lambda h\}. \end{aligned}\right] \tag{1.43}$$

It is impossible to make both the normal and the shear stresses simultaneously equal to zero for all (r, ϕ), for the B_1- or the B_2-potential. This is possible for a combination of these potentials if the following equations can be fulfilled

$$\left.\begin{aligned} &B_1^* \cos\lambda h - B_2^* \lambda h \sin\lambda h = 0, \\ &B_1^* \sin\lambda h + B_2^* \sin\lambda h + B_2^* \lambda h \cos\lambda h = 0. \end{aligned}\right] \tag{1.44}$$

This system has non-trivial solutions if /13

$$\begin{vmatrix} \cos\lambda h & -\lambda h \sin\lambda h \\ \sin\lambda h & \sin\lambda h + \lambda h \cos\lambda h \end{vmatrix} = 0, \tag{1.45}$$

or, after all operations are performed,

$$\sin 2\lambda h + 2\lambda h = 0. \tag{1.46}$$

Except for $\lambda = 0$, equation (1.46) has no real roots. It is apparent, however, that there is an infinite number of complex roots which here are called eigen values and which are numbered with the subscript ℓ. If λ_ℓ is a root of equation (1.46), then $-\lambda_\ell$, $\overline{\lambda_\ell}$, and $\overline{-\lambda_\ell}$ are also roots (the superimposed bar indicates the complex conjugate roots). Because of the requirement that the corrections to the stresses when $r \to \infty$ tend toward zero, only the eigen values λ_ℓ are taken into consideration. Therefore, it is assumed that

$$\operatorname{Re} \lambda_l > 0. \tag{1.47}$$

Thus, by definition, $K_2(\lambda r)$ is (Ref. 19)

$$K_2(\lambda r) = \frac{\pi}{2} e^{\frac{3\pi}{2} i} H_2^{(1)}(i\lambda r), \tag{1.48}$$

while the asymptotic behavior of $H_2^{(1)}(u)$ is given (Ref. 19) by

$$H_2^{(1)}(u) \to \sqrt{\frac{2}{\pi u}} e^{i\left(u - \frac{5}{4\pi}\right)}. \tag{1.49}$$

The solution to expression (1.44) in terms of λ_ℓ is

$$\overset{*}{B}_{1l} = \overset{*}{B}_{2l} \lambda_l h \operatorname{tg} \lambda_l h, \tag{1.50}$$

by which combination (1.42) may be further specified

$$\left.\begin{aligned} B_1 &= \frac{1}{2} \frac{Ta^2}{2G} b_l \lambda_l h \operatorname{tg} \lambda_l h \frac{K_2(\lambda_l r)}{K_2(\lambda_l a)} \cos \lambda_l z \cos 2\varphi, \\ B_2 &= \frac{1}{2} \frac{Ta^2}{2G} b_l \frac{K_2(\lambda_l r)}{K_2(\lambda_l a)} \cos \lambda_l z \cos 2\varphi. \end{aligned}\right] \tag{1.51}$$

In this b_ℓ is an arbitrary complex constant. The (1.51) solution, which is complex, is taken together with solution /14

$$\left.\begin{aligned} \bar{B}_1 &= \frac{1}{2} \frac{Ta^2}{2G} \bar{b}_l \bar{\lambda}_l h \operatorname{tg} \bar{\lambda}_l h \frac{K_2(\bar{\lambda}_l r)}{K_2(\bar{\lambda}_l a)} \cos \bar{\lambda}_l z \cos 2\varphi, \\ \bar{B}_2 &= \frac{1}{2} \frac{Ta^2}{2G} \bar{b}_l \frac{K_2(\bar{\lambda}_l r)}{K_2(\bar{\lambda}_l a)} \cos \bar{\lambda}_l z \cos 2\varphi, \end{aligned}\right]$$

and this gives rise to a real result. Therefore, in the following λ_ℓ can always be assumed to mean an eigen value which also has a positive imaginary part.

For the further elaboration of this point, it is reasonable to make the quantities dimensionless. For this purpose we introduce

$$\left. r^* = \frac{r}{a} \right] \tag{1.52}$$

$$
\left.\begin{aligned}
z^* &= \frac{z}{h} \\
\lambda^* &= \lambda h
\end{aligned}\right]
$$

Since there is no fear of confusion, the asterisks in the symbols are omitted from now on. The total system of solutions with the eigen value equation is

$$
\left.\begin{aligned}
A &= \frac{T a^2}{2G} \sum_{k=1}^{\infty} a_k \frac{K_2(k\pi\beta r)}{K_2(k\pi\beta)} \cos k\pi z \sin 2\varphi, \\
B_1 &= \operatorname{Re} \frac{T a^2}{2G} \sum_{l=1}^{\infty} b_l \lambda_l \operatorname{tg} \lambda_l \frac{K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)} \cos \lambda_l z \cos 2\varphi, \\
B_2 &= \operatorname{Re} \frac{T a^2}{2G} \sum_{l=1}^{\infty} b_l \frac{K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)} \cos \lambda_l z \cos 2\varphi, \\
&\sin 2\lambda_l + 2\lambda_l = 0,
\end{aligned}\right] \tag{1.53}
$$

in which the dimensionless parameter β is defined by /15

$$
\beta = \frac{a}{h}. \tag{1.54}
$$

In the next step, an attempt will be made to define constants (a_k, b_ℓ) in such a way that the stress system derived from expression (1.53), together with expression (1.31), fulfills the boundary values for all values of z when r = 1.

1.7 Stresses on the Edge of Hole when r = 1

As has already been mentioned previously, the stresses may be written in the form

$$
\left.\begin{aligned}
\sigma_r &= \sigma_r^{(0)} + \sigma_r^{(2)} \cos 2\varphi, \\
\tau_{rz} &= \tau_{rz}^{(2)} \cos 2\varphi, \\
\tau_{r\varphi} &= \tau_{r\varphi}^{(0)} + \tau_{r\varphi}^{(2)} \sin 2\varphi,
\end{aligned}\right] \tag{1.55}
$$

in which the stress values provided with a superscript are independent of ϕ. According to expression (1.33), it thus follows for the stresses, when r = 1, that

$$
\left.\begin{aligned}
\frac{\sigma_r^{(2)}}{T} &= \frac{1}{2} + 6C - D\left[4(1+\nu) + \frac{12\nu}{\beta^2} z^2\right], \\
\tau_{rz}^{(2)} &= 0, \\
\frac{\tau_{r\varphi}^{(2)}}{T} &= -\frac{1}{2} + 6C - D\left[2(1+\nu) + \frac{12\nu}{\beta^2} z^2\right].
\end{aligned}\right] \tag{1.56}
$$

By means of the well-known expansion (Ref. 21)

$$z^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k \cos k\pi z}{k^2} \quad (-1 \leq z \leq 1) \tag{1.57}$$

we may write

/16

$$\left.\begin{aligned}
\frac{\sigma_r^{(2)}}{T} &= \frac{1}{2} + 6\,C - D \Big\{ 4(1+\nu) + \\
&\quad + \frac{4\nu}{\beta^2} + \frac{48\nu}{\pi^2\beta^2} \sum_{k=1}^{\infty} \frac{(-1)^k \cos k\pi z}{k^2} \Big\}, \\
\tau_{rz}^{(2)} &= 0, \\
\frac{\tau_{r\varphi}^{(2)}}{T} &= -\frac{1}{2} + 6\,C - D \Big\{ 2(1+\nu) - \\
&\quad - \frac{4\nu}{\beta^2} + \frac{48\nu}{\pi^2\beta^2} \sum_{k=1}^{\infty} \frac{(-1)^k \cos k\pi z}{k^2} \Big\}.
\end{aligned}\right] \tag{1.58}$$

Use is made of certain well-known recurrence relationships for the Bessel functions (Ref. 20)

$$K'_2(\lambda) = K_1(\lambda) - \frac{2}{\lambda} K_2(\lambda), \tag{1.59}$$

$$K''_2(\lambda) = -\frac{1}{\lambda} K_1(\lambda) + K_2(\lambda) + \frac{6}{\lambda^2} K_2(\lambda). \tag{1.60}$$

to derive the stress magnitudes from the potentials. By means of formulas (1.18), we derive

$$\left.\begin{aligned}
\frac{\sigma_r^{(2)}}{T} &= \sum_{k=1}^{\infty} a_k \{2\,K(k\pi\beta) - 2\} \cos k\pi z, \\
\frac{\tau_{rz}^{(2)}}{T} &= \beta \sum_{k=1}^{\infty} a_k \{-k\pi\} \sin k\pi z, \\
\frac{\tau_{r\varphi}^{(2)}}{T} &= \sum_{k=1}^{\infty} a_k \Big\{ K(k\pi\beta) - \frac{k^2\pi^2\beta^2}{2} - 4 \Big\} \cos k\pi z.
\end{aligned}\right] \tag{1.61}$$

from potential A by expression (1.53) when r = 1. Here a new symbol for the quotient of Bessel functions $K'_2(\lambda)$ and $K_2(\lambda)$ is introduced, given by formula

$$K(\lambda) = \frac{\lambda K'_2(\lambda)}{K_2(\lambda)} = \frac{\lambda K_1(\lambda)}{K_2(\lambda)} - 2. \tag{1.62}$$

This K-function plays an important role throughout the entire work. /17

Subsequently, from potential B_1 by means of expression (1.19), the following formulas

$$\frac{\sigma_r^{(2)}}{T} = \mathrm{Re} \sum_{l=1}^{\infty} b_l \lambda_l \,\mathrm{tg}\, \lambda_l \{-K(\lambda_l \beta) + \lambda_l^2 \beta^2 + 4\} \cos \lambda_l z, \Big] \tag{1.63}$$

$$\left.\begin{aligned}
\frac{\tau_{rz}^{(2)}}{T} &= \beta \operatorname{Re} \sum_{l=1}^{\infty} b_l \lambda_l \operatorname{tg} \lambda_l \{-\lambda_l K(\lambda_l \beta)\} \sin \lambda_l z, \\
\frac{\tau_{r\varphi}^{(2)}}{T} &= \operatorname{Re} \sum_{l=1}^{\infty} b_l \lambda_l \operatorname{tg} \lambda_l \{-2K(\lambda_l \beta) + 2\} \cos \lambda_l z,
\end{aligned}\right]$$

are derived for the stresses when r = 1. From B_2, by expression (1.20) and for r = 1 we can compute

$$\left.\begin{aligned}
\frac{\sigma_r^{(2)}}{T} &= \operatorname{Re} \Bigg[\sum_{l=1}^{\infty} b_l \{-2(1-\nu)(K(\lambda_l \beta) - 4) + \\
&+ 2\lambda_l^2 \beta^2\} \cos \lambda_l z + \sum_{l=1}^{\infty} b_l \{K(\lambda_l \beta) - \lambda_l^2 \beta^2 - 4\} \lambda_l z \sin \lambda_l z \Bigg], \\
\frac{\tau_{rz}^{(2)}}{T} &= \beta \operatorname{Re} \Bigg[\sum_{l=1}^{\infty} b_l \{-\lambda_l K(\lambda_l \beta)\} \sin \lambda_l z + \\
&+ \sum_{l=1}^{\infty} b_l \{-\lambda_l K(\lambda_l \beta)\} \lambda_l z \cos \lambda_l z \Bigg], \\
\frac{\tau_{r\varphi}^{(2)}}{T} &= \operatorname{Re} \Bigg[\sum_{l=1}^{\infty} b_l \{-4(1-\nu)(K(\lambda_l \beta) - 1)\} \cos \lambda_l z + \\
&+ \sum_{l=1}^{\infty} b_l \{2(K(\lambda_l \beta) - 1)\} \lambda_l z \sin \lambda_l z \Bigg].
\end{aligned}\right] \quad (1.64)$$

Stresses (1.61), (1.63) and (1.64) are taken together. The result is written /18

$$\left.\begin{aligned}
\frac{\sigma_r^{(2)}}{T} &= \operatorname{Re} \Bigg[\sum_{k=1}^{\infty} a_k p_k^{(1)} \cos k\pi z + \\
&+ \sum_{l=1}^{\infty} b_l q_l^{(1)} \cos \lambda_l z + \sum_{l=1}^{\infty} b_l r_l^{(1)} \lambda_l z \sin \lambda_l z \Bigg], \\
\frac{\tau_{rz}^{(2)}}{T} &= -\beta \operatorname{Re} \Bigg[\sum_{k=1}^{\infty} a_k p_k^{(2)} \sin k\pi z + \\
&+ \sum_{l=1}^{\infty} b_l q_l^{(2)} \sin \lambda_l z + \sum_{l=1}^{\infty} b_l r_l^{(2)} \lambda_l z \cos \lambda_l z \Bigg], \\
\frac{\tau_{r\varphi}^{(2)}}{T} &= \operatorname{Re} \Bigg[\sum_{k=1}^{\infty} a_k p_k^{(3)} \cos k\pi z + \\
&+ \sum_{l=1}^{\infty} b_l q_l^{(3)} \cos \lambda_l z + \sum_{l=1}^{\infty} b_l r_l^{(3)} \lambda_l z \sin \lambda_l z \Bigg],
\end{aligned}\right] \quad (1.65)$$

where the following values are introduced

$$\left.\begin{aligned}
p_k^{(1)} &= 2K(k\pi\beta) - 2, \\
p_k^{(2)} &= k\pi, \\
p_k^{(3)} &= K(k\pi\beta) - \frac{(k\pi\beta)^2}{2} - 4;
\end{aligned}\right] \quad (1.66)$$

$$\left.\begin{aligned} q_l^{(1)} &= \{\lambda_l \operatorname{tg} \lambda_l + 2(1-\nu)\} \{-K(\lambda_l \beta) + 4\} + \\ &\qquad + \lambda^2_l \beta^2 (\lambda_l \operatorname{tg} \lambda_l + 2), \\ q_l^{(2)} &= \lambda_l (\lambda_l \operatorname{tg} \lambda_l + 1) \{K(\lambda_l \beta)\}, \\ q_l^{(3)} &= -2(\lambda_l \operatorname{tg} \lambda_l + 2(1-\nu)) (K(\lambda_l \beta) - 1); \end{aligned}\right] \tag{1.67}$$

$$\left.\begin{aligned} r_l^{(1)} &= K(\lambda_l \beta) - \lambda^2_l \beta^2 - 4, \\ r_l^{(2)} &= \lambda_l K(\lambda_l \beta), \\ r_l^{(3)} &= 2(K(\lambda_l \beta) - 1). \end{aligned}\right] \tag{1.68}$$

The form of the stresses, as given in expression (1.65), is still in- /19
convenient for further development. Therefore, these stresses will be expanded in Fourier series. It is simple to derive the following auxilliary formulas for real or complex λ_ℓ and for $-1 \leq z \leq 1$

$$\left.\begin{aligned} \cos \lambda_l z &= s_{lo} + \sum_{k=1}^{\infty} s_{lk} \cos k\pi z, \\ \sin \lambda_l z &= \sum_{k=1}^{\infty} u_{lk} \sin k\pi z, \\ \lambda_l z \sin \lambda_l z &= t_{lo} + \sum_{k=1}^{\infty} t_{lk} \cos k\pi z, \\ \lambda_l z \cos \lambda_l z &= \sum_{k=1}^{\infty} v_{lk} \sin k\pi z, \end{aligned}\right] \tag{1.69}$$

in which the following notation is employed:

$$\left.\begin{aligned} s_{lo} &= \frac{\sin \lambda_l}{\lambda_l}, \\ s_{lk} &= 2\lambda_l \sin \lambda_l \frac{(-1)^k}{\lambda^2_l - k^2 \pi^2}, \\ u_{lk} &= 2 \sin \lambda_l \frac{(-1)^k k\pi}{\lambda^2_l - k^2 \pi^2}, \\ t_{lo} &= -\cos \lambda_l + \frac{\sin \lambda_l}{\lambda_l}, \\ t_{lk} &= 2\lambda_l \sin \lambda_l (-1)^k \left[\frac{\lambda^2_l - k^2\pi^2}{(\lambda^2_l - k^2 \pi^2)^2} - \frac{\lambda_l \cot \lambda_l}{\lambda^2_l - k^2 \pi^2}\right], \\ v_{lk} &= 2\lambda_l \sin \lambda_l (-1)^k \left[\frac{k\pi \cot \lambda_l}{\lambda^2_l - k^2 \pi^2} - \frac{2\lambda_l k\pi}{(\lambda^2_l - k^2 \pi^2)^2}\right]. \end{aligned}\right] \tag{1.70}$$

It may be noted that these coefficients are, on the whole, asymmetrical in their subscripts.

If expression (1.70) is introduced into expression (1.65), we have /20

$$
\left.
\begin{aligned}
\frac{\sigma_r^{(2)}}{T} &= \mathrm{Re}\left[\sum_{l=1}^{\infty} b_l \psi_{lo}^{(1)} + \sum_{k=1}^{\infty}\left\{ a_k p_k^{(1)} + \sum_{l=1}^{\infty} b_l \psi_{lk}^{(1)} \right\}\cos k\pi z\right]; \\
\frac{\tau_{rz}^{(2)}}{T} &= -\beta\,\mathrm{Re}\left[\sum_{k=1}^{\infty}\left\{ a_k p_k^{(2)} + \sum_{l=1}^{\infty} b_l \psi_{lk}^{(2)} \right\}\sin k\pi z\right], \\
\frac{\tau_{r\varphi}^{(2)}}{T} &= \mathrm{Re}\left[\sum_{l=1}^{\infty} b_l \psi_{lo}^{(3)} + \sum_{k=1}^{\infty}\left\{ a_k p_k^{(3)} + \sum_{l=1}^{\infty} b_l \psi_{lk}^{(3)} \right\}\cos k\pi z\right].
\end{aligned}
\right] \tag{1.71}
$$

Here abbreviations given by

$$
\left.
\begin{aligned}
\psi_{lk}^{(1)} &= q_l^{(1)} s_{lk} + r_l^{(1)} t_{lk}, \\
\psi_{lk}^{(2)} &= q_l^{(2)} u_{lk} + r_l^{(2)} v_{lk}, \\
\psi_{lk}^{(3)} &= q_l^{(3)} s_{lk} + r_l^{(3)} t_{lk};
\end{aligned}
\right] \tag{1.72}
$$

are again introduced. These formulas hold for (k = 0, 1, 2, ...), (ℓ = 1, 2, ...) on the condition that $\psi_{\ell o}^{(2)} = 0$.

The formal solution of the problem is obtained by superposing the stresses according to expression (1.71) and those according to expression (1.58), and by setting the coefficients of cos kπz and sin kπz one after another equal to zero in the development. In this way, we obtain a system of equations of the following form

$$
\left.
\begin{aligned}
\mathrm{Re}\sum_{l=1}^{\infty} b_l \psi_{lo}^{(1)} + 6C - D\left\{4(1+\nu) + \frac{4\nu}{\beta^2}\right\} &= -\frac{1}{2}, \\
\mathrm{Re}\sum_{l=1}^{\infty} b_l \psi_{lo}^{(3)} + 6C - D\left\{2(1+\nu) + \frac{4\nu}{\beta^2}\right\} &= \frac{1}{2},
\end{aligned}
\right] \tag{1.73}
$$

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$$
\left.
\begin{aligned}
a_k p_k^{(1)} + \mathrm{Re}\sum_{l=1}^{\infty} b_l \psi_{lk}^{(1)} &= D\,\frac{48\nu}{\pi^2\beta^2}\,\frac{(-1)^k}{k^2}; \\
a_k p_k^{(2)} + \mathrm{Re}\sum_{l=1}^{\infty} b_l \psi_{lk}^{(2)} &= 0, \\
a_k p_k^{(3)} + \mathrm{Re}\sum_{l=1}^{\infty} b_l \psi_{lk}^{(3)} &= D\,\frac{48\nu}{\pi^2\beta^2}\,\frac{(-1)^k}{k^2}. \\
&(k = 1, 2, \ldots.)
\end{aligned}
\right] \tag{1.74}
$$

The system (1.74) is a three-dimensional infinite system of equations with a three-dimensional infinite number of real unknowns, since each b_ℓ counts twice. If this system has a solution, a_k and b_ℓ are then expressed in terms of D. By means of expression (1.73), C and D may then be found afterwards. It may be easily proven that system (1.74) may be solved. Therefore, a few remarks will suffice, in which the assumption is made that this system always has one solution.

To obtain practical results, we must approximate system (1.74) by a system of 3N unknowns: $a_1, \ldots, a_N; b_1, \ldots, b_N$. The solution of the reduced system is then only a suitable approximation of the solution of the given finite system if two conditions are fulfilled. In the first place, the values of a_k, b_ℓ, computed from the reduced system, must be sufficiently *stable*, i.e., they may vary only slightly when the number of equations and unknowns taken into consideration is increased. Further, the calculated values of a_k and b_ℓ must diminish with sufficient speed as the subscripts k and ℓ grow larger, with the result that the calculated stresses *converge* well. The extent to which these conditions are fulfilled can only be ascertained by carrying out the calculations.

The accuracy attainable by reducing the system (1.74) is greatly dependent on the parameter β, which is expressed in the coefficients of the system. This is physically clear. For a very thin plate ($\beta \to \infty$) the solution of system (1.74) is assumed to be $a_k \to 0$, $b_\ell \to 0$, so that from expression (1.73) the values in expression (1.34a) of the plane state of stress follow for C and D. With great, but finite values of β, an asymptotic development is possible in which it turns out that all unknowns a_k, B_ℓ are of order β^{-4}. Numerically, /22 it will be found that a_1 and b_1 greatly predominate (Section 2.1) so that N = 1 can be satisfied. The physical meaning of this lies in the fact that the deviation from the generalized plane state of stress for a thin plate is well approximated by the first term of the Fourier series in the thickness direction. For thicker plates, a larger number of equations and unknowns must be used.

1.8 Derivation of the Coefficients

Before proceeding to a general examination of equations (1.73) and (1.74), we find it desirable to take a closer look at the coefficients which occur. First a few simple identities are given. Thus, it is assumed that

$$\left.\begin{aligned} t_{lk} &= s_{lk}\left[1 - \lambda_l \cot \lambda_l + \frac{2\,k^2\pi^2}{\lambda_l^2 - k^2\pi^2}\right], \\ \lambda_l \operatorname{tg} \lambda_l &+ \lambda_l \cot \lambda_l = -1, \\ v_{lk} &= u_{lk}\left[\lambda_l \cot \lambda_l - \frac{2\,\lambda_l^2}{\lambda_l^2 - k^2\pi^2}\right]. \end{aligned}\right] \tag{1.75}$$

Further, we may write

$$
\left.\begin{aligned}
q^{(1)}_l &= -r^{(1)}_l\left[\lambda_l \operatorname{tg}\lambda_l + 2(1-\nu)\right] + 2\nu\lambda^2_l\beta^2, \\
q^{(2)}_l &= r^{(2)}_l\left[\lambda_l \operatorname{tg}\lambda_l + 1\right], \\
q^{(3)}_l &= -r^{(3)}_l\left[\lambda_l \operatorname{tg}\lambda_l + 2(1-\nu)\right].
\end{aligned}\right] \tag{1.76}
$$

By means of relationships (1.75) and (1.76), from expression (1.72) we find

$$
\left.\begin{aligned}
\psi^{(1)}_{lk} &= -4\beta^2\lambda^3_l \sin\lambda_l (-1)^k \frac{k^2\pi^2}{(\lambda^2_l - k^2\pi^2)^2} + \\
&+ 4\lambda_l \sin\lambda_l \frac{(-1)^k}{\lambda^2_l - k^2\pi^2}\left\{K(\lambda_l\beta) - 4\left\{\nu + \frac{k^2\pi^2}{(\lambda^2_l - k^2\pi^2)}\right\}\right\},
\end{aligned}\right] \tag{1.77}
$$

$$
\psi^{(1)}_{lo} = 2\nu\frac{\sin\lambda_l}{\lambda_l}\{K(\lambda_l\beta) - 4\}, \tag{1.78}
$$

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$$
\psi^{(2)}_{lk} = -4(-1)^k\lambda^3_l \sin\lambda_l \frac{k\pi}{(\lambda^2_l - k^2\pi^2)^2} K(\lambda_l\beta), \tag{1.79}
$$

$$
\left.\begin{aligned}
\psi^{(3)}_{lk} &= 8(-1)^k\lambda_l \sin\lambda_l \frac{1}{\lambda^2_l - k^2\pi^2}\left\{\nu + \right. \\
&\left. + \frac{k^2\pi^2}{\lambda^2_l - k^2\pi^2}\right\}\{K(\lambda_l\beta) - 1\},
\end{aligned}\right] \tag{1.80}
$$

$$
\psi^{(3)}_{lo} = 4\nu\frac{\sin\lambda_l}{\lambda_l}\{K(\lambda_l\beta) - 1\}. \tag{1.81}
$$

Then from system (1.74) we eliminate a_k:

$$
a_k = -\frac{1}{p^{(2)}_k}\operatorname{Re}\sum_{l=1}^{\infty} b_l\psi^{(2)}_{lk}, \tag{1.82}
$$

so that this system is converted into

$$
\left.\begin{aligned}
\operatorname{Re}\sum_{l=1}^{\infty} b_l\chi^{(1)}_{lk} &= D\frac{48\nu}{\pi^2\beta^2}\frac{(-1)^k}{k^2}, \\
\operatorname{Re}\sum_{l=1}^{\infty} b_l\chi^{(2)}_{lk} &= D\frac{48\nu}{\pi^2\beta^4}\frac{(-1)^k}{k^2}, \\
&(k = 1, 2, \ldots)
\end{aligned}\right] \tag{1.83}
$$

with

$$
\left.\begin{aligned}
\chi^{(1)}_{lk} &= \psi^{(1)}_{lk} - \frac{p^{(1)}_k}{p^{(2)}_k}\psi^{(2)}_{lk}, \\
\chi^{(2)}_{lk} &= \psi^{(3)}_{lk} - \frac{p^{(3)}_k}{p^{(2)}_k}\psi^{(2)}_{lk}.
\end{aligned}\right] \begin{aligned}&(k = 1, 2\ldots)\\ &(l = 1, 2\ldots)\end{aligned} \tag{1.84}
$$

The last simplifications are obtained by setting

$$b_l = D \frac{48\nu}{\pi^2 \beta^2} \frac{c_l}{4 \lambda^3_l \sin \lambda_l}, \tag{1.85}$$

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$$\chi^{*(n)}_{lk} = \frac{\chi^{(n)}_{lk} (-1)^k}{4 \lambda^3_l \sin \lambda_l}, \tag{1.86}$$

whereby expression (1.83) becomes

$$\left.\begin{aligned} \operatorname{Re} \sum_{l=1}^{\infty} c_l \chi^{*(1)}_{lk} = \frac{1}{k^2}, \\ \operatorname{Re} \sum_{l=1}^{\infty} c_l \chi^{*(2)}_{lk} = \frac{1}{k^2}. \end{aligned}\right] (k = 1, 2 \ldots) \tag{1.87}$$

Here c_ℓ is independent of D. If the two equations (1.73) are subtracted from each other, we obtain an equation for D. It is found that

$$D = \frac{1}{2(1+\nu) - \frac{12\nu}{\pi^2 \beta^2} \operatorname{Re} \sum_{l=1}^{\infty} \frac{c_l}{\lambda^3_l \sin \lambda_l} (\psi^{(1)}_{lo} - \psi^{(3)}_{lo})}. \tag{1.88}$$

By means of equations (1.88), (1.85), (1.78), and (1.81), finally b_ℓ is found:

$$b_l = \frac{c_l / 4 \lambda^3_l \sin \lambda_l}{\frac{\pi^2 (1+\nu)}{24\nu} \beta^2 + \frac{\nu}{2} \beta \operatorname{Re} \sum_{l=1}^{\infty} c_l \frac{K_1(\lambda_l \beta)}{K_2(\lambda_l \beta)} \frac{1}{\lambda^3_l}}, \tag{1.89}$$

while a_k is defined by equations (1.89) and (1.82). The abbreviation

$$d_l = 4\lambda^3_l \sin \lambda_l \, . \, b_l = \frac{c_l}{\frac{\pi^2 (1+\nu)}{24\nu} \beta^2 + \frac{\nu}{2} \beta \operatorname{Re} \sum_{l=1}^{\infty} c_l \frac{K_1(\lambda_l \beta)}{K_2(\lambda_l \beta)} \frac{1}{\lambda^3_l}}. \tag{1.90}$$

still plays a role in the expressions for the stresses.

1.9 The Thin Plate

The coefficients of systems (1.73) and (1.74) are complicated functions of parameter β, and it seems reasonable to investigate whether in extreme cases the system can be simplified. As already noted above, for the thin plate /25 ($h \to 0$, i.e., $\beta \to \infty$) the system will have the solution $a_k = b_\ell = 0$, while C and D alone will not equal zero (plane state of stress). The state of stress of a thin plate, for which h has a finite value, may now be regarded as a perturbation of this plane state of stress. Therefore, first of all we shall seek the solution of systems (1.73) and (1.74) which may be written as an expansion in powers of $\frac{1}{\beta}$. For this purpose, use is made of the following well-known asymptotic expansions [see (Ref. 20), page 374]

$$K_1(z) = -\sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{3}{8z} - \frac{15}{128 z^2} + \ldots \right], \tag{1.91}$$

$$K_2(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left[1 + \frac{15}{8z} + \frac{105}{128 z^2} + \dots\right], \tag{1.91}$$

which are important when $|z|$ is large and $|\arg z| < \frac{3}{2}\pi$. From expression (1.91) we derive

$$K(z) = -z - \frac{1}{2} - \frac{15}{8}\,\frac{1}{z} + \dots \quad (|z| >> 1) \tag{1.92}$$

By means of this relationship expression (1.66) is written as

$$\left.\begin{aligned} p_k^{(1)} &= -2k\pi\beta - 3 - \frac{15}{4}\,\frac{1}{k\pi\beta} + \dots, \\ p_k^{(2)} &= k\pi, \\ p_k^{(3)} &= -\frac{(k\pi\beta)^2}{2} - k\pi\beta - \frac{9}{2} + \dots\dots \end{aligned}\right] (\beta >> 1) \tag{1.93}$$

For the $\psi_{\ell k}$ coefficients, the following asymptotic series hold /26

$$\left.\begin{aligned} \psi_{lk}^{(1)} &= \beta^2\left[-4(-1)^k \lambda^3_l \sin\lambda_l \frac{k^2\pi^2}{(\lambda^2_l - k^2\pi^2)^2}\right] - \\ &- \beta\left[4(-1)^k\lambda^2_l \sin\lambda_l \left(\frac{1}{\lambda^2_l - k^2\pi^2}\right)\left(\nu + \frac{k^2\pi^2}{\lambda^2_l - k^2\pi^2}\right)\right] - \\ &- 18(-1)^k \lambda_l \sin\lambda_l \frac{1}{\lambda^2_l - k^2\pi^2}\cdot\left(\nu + \frac{k^2\pi^2}{\lambda^2_l - k^2\pi^2}\right) + \dots\dots \\ \psi_{lo}^{(1)} &= -\beta.\ 2\nu\sin\lambda_l - 9\nu\frac{\sin\lambda_l}{\lambda_l} + \dots, \\ \psi_{lk}^{(2)} &= \beta\left[4(-1)^k\lambda^4_l \sin\lambda_l \frac{k\pi}{(\lambda^2_l - k^2\pi^2)^2}\right] + \\ &\qquad + 2(-1)^k\lambda^3_l \sin\lambda_l \frac{k\pi}{(\lambda^2_l - k^2\pi^2)^2} + \dots, \\ \psi_{lk}^{(3)} &= -\beta\left[8(-1)^k\lambda^2_l \sin\lambda_l \left(\frac{1}{\lambda^2_l - k^2\pi^2}\right)\left(\nu + \frac{k^2\pi^2}{\lambda^2_l - k^2\pi^2}\right)\right] \\ &- 12(-1)^k\lambda_l \sin\lambda_l \left(\frac{1}{\lambda^2_l - k^2\pi^2}\right)\left(\nu + \frac{k^2\pi^2}{\lambda^2_l - k^2\pi^2}\right) + \dots, \\ \psi_{lo}^{(3)} &= -\beta.\ 4\nu\sin\lambda_l - 6\nu\frac{\sin\lambda_l}{\lambda_l} + \dots\dots \quad (\beta >> 1) \end{aligned}\right] \tag{1.94}$$

The desired asymptotic expansion is written in the following form

$$\left.\begin{aligned} C &= C^{(0)} + \frac{C^{(1)}}{\beta} + \dots, \\ D &= D^{(0)} + \frac{D^{(1)}}{\beta} + \dots, \end{aligned}\right] \quad (\beta >> 1) \tag{1.95}$$

$$\left.\begin{aligned} a_k &= a_k^{(0)} + \frac{a_k^{(1)}}{\beta} + \ldots, \\ b_l &= b_l^{(0)} + \frac{b_l^{(1)}}{\beta} + \ldots, \end{aligned}\right] \quad (1.95)$$

and then, together with expansions (1.93) and (1.94), is substituted in equations (1.73) and (1.74). If the right sides of these equations are transposed to the left side, expansions in powers of $\frac{1}{\beta}$ result, which, when set equal to zero, result in a sufficient number of equations to determine the coefficients in expression (1.95). In this fashion we find /27

$$\left.\begin{aligned} & C^{(0)} = \frac{1}{4}, \; D^{(0)} = \frac{1}{2(1+\nu)}, \\ & C^{(1)} = 0, \; D^{(1)} = 0, \\ & C^{(2)} = \frac{2}{3}\nu. \; D^{(0)} = \frac{\nu}{3(1+\nu)}, \; D^{(2)} = 0, \\ & a_k^{(0)} = a_k^{(1)} = a_k^{(2)} = a_k^{(3)} = 0, \\ & b_l^{(0)} = b_l^{(1)} = b_l^{(2)} = b_l^{(3)} = 0, \end{aligned}\right] \quad (1.96)$$

as well as

$$a_k^{(4)} = -\frac{48\nu}{1+\nu} \frac{(-1)^k}{k^4 \pi^4}. \quad (1.97)$$

The coefficient $b_\ell^{(4)}$ is solved by introducing the abbreviation

$$b_l^{(4)} = -\frac{6\nu}{1+\nu} \frac{c_l^{(4)}}{\lambda^3{}_l \sin \lambda_l}, \quad (1.98)$$

from system

$$\left.\begin{aligned} & \operatorname{Re} \sum_{l=1}^{\infty} c_l^{(4)} \frac{1}{(\lambda^2{}_l - k^2\pi^2)^2} = \frac{1}{k^4 \pi^4}, \\ & \operatorname{Re} \sum_{l=1}^{\infty} c_l^{(4)} \frac{\lambda_l}{(\lambda^2{}_l - k^2\pi^2)^2} = 0. \end{aligned}\right] (k = 1, 2, \ldots) \quad (1.99)$$

The first approximation is completed with the solution of these equations. We may proceed in the same manner when computing higher approximations. The following system /28

$$\left.\begin{aligned} & 6\,C^{(3)} - 4(1+\nu)\,D^{(3)} = \operatorname{Re} \sum_{l=1}^{\infty} b_l^{(4)} \,.\, 2\nu \sin \lambda_l, \\ & 6\,C^{(3)} - 2(1+\nu)\,D^{(3)} = \operatorname{Re} \sum_{l=1}^{\infty} b_l^{(4)} \,.\, 4\nu \sin \lambda_l. \end{aligned}\right] \quad (1.100)$$

is found for $C^{(3)}$ and $D^{(3)}$ from expression (1.73).

For $a_k^{(5)}$ we find

$$a_k^{(5)} = -\frac{2}{k\pi}\, a_k^{(4)} - \frac{16(-1)^k}{k^2\pi^2}\, \mathrm{Re} \sum_{l=1}^{\infty} b_l^{(4)} \lambda_l^2 \sin\lambda_l \left(\frac{1}{\lambda_l^2 - k^2\pi^2}\right)\left(\nu + \frac{k^2\pi^2}{\lambda_l^2 - k^2\pi^2}\right), \tag{1.101}$$

while $b_\ell^{(5)}$ with the abbreviation

$$b_l^{(5)} = \frac{6\nu}{1+\nu}\, \frac{c_l^{(5)}}{\lambda_l^3 \sin\lambda_l}, \tag{1.102}$$

is calculated from

$$\begin{aligned}
&\mathrm{Re} \sum_{l=1}^{\infty} c_l^{(5)} \frac{1}{(\lambda_l^2 - k^2\pi^2)^2} = \frac{4}{k^5\pi^5} + \\
&+ \frac{1}{k^2\pi^2}\, \mathrm{Re} \sum_{l=1}^{\infty} \frac{c_l^{(4)}}{\lambda_l}\left(\frac{1}{\lambda_l^2 - k^2\pi^2}\right)\left(\nu + \frac{k^2\pi^2}{\lambda_l^2 - k^2\pi^2}\right), \\
&\mathrm{Re} \sum_{l=1}^{\infty} c_l^{(5)} \frac{\lambda_l}{(\lambda_l^2 - k^2\pi^2)^2} = \frac{2}{k^4\pi^4} + \\
&+ \frac{1}{2}\, \mathrm{Re} \sum_{l=1}^{\infty} c_l^{(4)} \frac{1}{(\lambda_l^2 - k^2\pi^2)^2} = \frac{5}{2}\, \frac{1}{k^4\pi^4}. \\
&(k = 1, 2, \ldots.)
\end{aligned} \tag{1.103}$$

In this ~~way, ever higher order terms are~~ obtained, but there is little /29 practical use in so doing because of the asymptotic nature of the expansion. Better results may be obtained by proceeding directly from equations (1.73) and (1.74) for finite values of the ratio of hole diameter to plate thickness.

1.10 Stresses

If coefficients C, D, a_k and b_ℓ are known, the stress values are determined from the potentials by means of equations (1.18), (1.19), and (1.20) and added to solution (1.33). After some calculations, the following expression is found for stress σ_r

$$\begin{aligned}
\frac{\sigma_r}{T} = &\frac{1}{2}\left(1 - \frac{1}{r^2}\right) + \cos 2\varphi \left[\frac{1}{2} + \right. \\
&+ 6C\frac{1}{r^4} - D\left\{\frac{4(1+\nu)}{r^2} + \frac{12\nu}{\beta^2}\, \frac{z^2}{r^4}\right\} - \\
&- \frac{2}{r^2} \sum_{k=1}^{\infty} a_k \cos k\pi z \left\{\frac{3\, K_2(k\pi\beta r)}{K_2(k\pi\beta)} - (k\pi\beta r)\, \frac{K_1(k\pi\beta r)}{K_2(k\pi\beta)}\right\} + \\
&+ \mathrm{Re} \sum_{l=1}^{\infty} b_l \left\{\frac{1}{r^2}\, \frac{6\, K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)} - \frac{\lambda_l \beta}{r}\, \frac{K_1(\lambda_l \beta r)}{K_2(\lambda_l \beta)} + \right. \\
&+ \left. \lambda_l^2 \beta^2 \frac{K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)}\right\} \{(\lambda_l \operatorname{tg} \lambda_l + 2(1-\nu))\} \cos \lambda_l z -
\end{aligned} \tag{1.104}$$

$$-\lambda_l z \sin \lambda_l z\} + 2\nu \operatorname{Re} \sum_{l=1}^{\infty} b_l \beta^2 \lambda^2_l \frac{K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)} \cos \lambda_l z \Bigg]. \Bigg] \tag{1.104}$$

As is easily verified, $\sigma_r = 0$ when $r = 1$. When $z = 0$ or $z = 1$, the formula does not become much simpler.

For stress σ_ϕ we can derive /30

$$\begin{aligned} \frac{\sigma_\varphi}{T} = \frac{1}{2}\left(1 + \frac{1}{r^2}\right) - \cos 2\varphi \Bigg[\frac{1}{2} + 6C \frac{1}{r^4} - \\ - D \frac{12\nu}{\beta^2} \frac{z^2}{r^4} + \frac{2}{r^2} \sum_{k=1}^{\infty} a_k \cos k\pi z \Bigg\{ \frac{3 K_2(k\pi\beta r)}{K_2(k\pi\beta)} - \\ - (k\pi\beta r) \frac{K_1(k\pi\beta r)}{K_2(k\pi\beta)} \Bigg\} + \operatorname{Re} \sum_{l=1}^{\infty} b_l \Bigg\{ \frac{1}{r^2} \frac{6 K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)} - \\ - \frac{\lambda_l \beta}{r} \frac{K_1(\lambda_l \beta r)}{K_2(\lambda_l \beta)} \Bigg\} \{-(\lambda_l \operatorname{tg} \lambda_l + 2) \cos \lambda_l z + \\ + \lambda_l z \sin \lambda_l z\} + 2\nu \operatorname{Re} \sum_{l=1}^{\infty} b_l \cos \lambda_l z \Bigg\{ \frac{1}{r^2} \frac{6 K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)} - \\ - \frac{\lambda_l \beta}{r} \frac{K_1(\lambda_l \beta r)}{K_2(\lambda_l \beta)} + \lambda^2_l \beta^2 \frac{K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)} \Bigg\} \Bigg]. \end{aligned} \tag{1.105}$$

When $r = 1$ this form is simplified to

$$\begin{aligned} \left(\frac{\sigma_\varphi}{T}\right)_{r=1} = 1 + \cos 2\varphi \Bigg[-4(1+\nu) D + \\ + \beta^2 \operatorname{Re} \sum_{l=1}^{\infty} b_l \lambda^3_l \operatorname{tg} \lambda_l \cos \lambda_l z + 2(1+\nu)\beta^2 . \\ \operatorname{Re} \sum_{l=1}^{\infty} b_l \lambda^2_l \cos \lambda_l z - \beta^2 \operatorname{Re} \sum_{l=1}^{\infty} b_l \lambda^3_l z \sin \lambda_l z \Bigg]. \end{aligned} \tag{1.106}$$

When d_ℓ defined in expression (1.90) is introduced, this equation is to be written as

$$\begin{aligned} \left(\frac{\sigma_\varphi}{T}\right)_{r=1} = 1 + \cos 2\varphi \Bigg[-4(1+\nu) D + \\ + \frac{\beta^2}{4} \operatorname{Re} \sum_{l=1}^{\infty} d_l \left(\frac{\cos \lambda_l z}{\cos \lambda_l} - \frac{z \sin \lambda_l z}{\sin \lambda_l} \right) + \\ + \frac{1}{2}(1+\nu) \beta^2 \operatorname{Re} \sum_{l=1}^{\infty} d_l \frac{\cos \lambda_l z}{\lambda_l \sin \lambda_l} \Bigg]. \end{aligned} \tag{1.107}$$

From this we obtain the important value /31

$$\left(\frac{\sigma_\varphi}{T}\right)_{\substack{r=1 \\ z=1}} = 1 + \cos 2\varphi \Bigg[-4(1+\nu) D + \frac{1+\nu}{2} \beta^2 \operatorname{Re} \sum_{l=1}^{\infty} d_l \frac{\cot \lambda_l}{\lambda_l} \Bigg]. \Bigg] \tag{1.108}$$

Also of interest is the value averaged over the thickness

$$\left(\frac{\bar{\sigma}_\varphi}{T}\right)_{r=1} = 1 + \cos 2\varphi \left[-4(1+\nu)D + \right. \left. + \frac{\nu}{2} \beta^2 \operatorname{Re} \sum_{l=1}^{\infty} d_l \frac{1}{\lambda^2_l} \right]. \tag{1.109}$$

For stress σ_z we find

$$\frac{\sigma_z}{T} = \cos 2\varphi \left[\frac{\beta^2}{4} \operatorname{Re} \sum_{l=1}^{\infty} d_l \frac{K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)} \left\{ \frac{z \sin \lambda_l z}{\sin \lambda_l} - \frac{\cos \lambda_l z}{\cos \lambda_l} \right\} \right]. \tag{1.110}$$

As may easily be verified, $\sigma_z = 0$ when $|z| = 1$. If $r = 1$, then

$$\left(\frac{\sigma_z}{T}\right)_{r=1} = \cos 2\varphi \frac{\beta^2}{4} \operatorname{Re} \sum_{l=1}^{\infty} d_l \left(\frac{z \sin \lambda_l z}{\sin \lambda_l} - \frac{\cos \lambda_l z}{\cos \lambda_l} \right), \tag{1.111}$$

while the averaged value is

$$\left(\frac{\bar{\sigma}_z}{T}\right)_{r=1} = \cos 2\varphi \frac{\beta^2}{2} \operatorname{Re} \sum_{l=1}^{\infty} \frac{d_l}{\lambda^2_l}. \tag{1.112}$$

For a very *thin* plate, expressions (1.95) may be introduced into the stress formulas. If the expansion is not continued further than to terms of order $\frac{1}{\beta^2}$, the following forms arise /32

$$\left(\frac{\sigma_\varphi}{T}\right)_{\substack{r=1\\z=1}} = 1 + \cos 2\varphi \left[-2 - \frac{12\nu}{\beta^2} \operatorname{Re} \sum_{l=1}^{\infty} c_l^{(4)} \frac{\cot \lambda_l}{\lambda_l} \right] = 1 + \cos 2\varphi \left[-2 + \frac{4\nu}{\beta^2} \right], \tag{1.113}$$

$$\left(\frac{\bar{\sigma}_\varphi}{T}\right)_{r=1} = 1 + \cos 2\varphi \left[-2 - \frac{1}{\beta^2} \frac{12\nu^2}{1+\nu} \operatorname{Re} \sum_{l=1}^{\infty} c_l^{(4)} \frac{1}{\lambda^2_l} \right]. \tag{1.114}$$

Use is made of

$$\operatorname{Re} \sum_{l=1}^{\infty} c_l^{(4)} \frac{\cot \lambda_l}{\lambda_l} = -\frac{1}{3}. \tag{1.115}$$

when deriving expression (1.113). This relationship may be obtained by means of identity

$$\sum_{k=1}^{\infty} \frac{k^2\pi^2}{(\lambda^2_l - k^2\pi^2)^2} = \frac{1}{2 \sin^2 \lambda_l}, \tag{1.116}$$

which is derived from the first and third equations in expression (1.69) by taking z equal to 1 therein and by eliminating the series

$$\sum_{k=1}^{\infty} \frac{1}{\lambda^2_l - k^2 \pi^2}$$

From expression (1.116) it immediately follows that

$$\operatorname{Re} \sum_{k,\, l=1}^{\infty} b_l \,.\, -4\lambda^3_l \sin \lambda_l \frac{k^2 \pi^2}{(\lambda^2_l - k^2\pi^2)^2} =$$

$$= \operatorname{Re} \sum_{l=1}^{\infty} b_l \, 2\lambda^2_l \cos \lambda_l.$$

If the plate is very thin ($\beta \to \infty$) /33

$$= D^{(0)} \frac{48\nu}{\pi^2 \beta^4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{8\nu D^{(0)}}{\beta^4} = \frac{4\nu}{1+\nu} \frac{1}{\beta^4},$$

may be written for the left side of this equation by taking advantage of expressions (1.74) and (1.94), while the right side equals

$$\operatorname{Re}\left[-\frac{12\nu}{1+\nu} \frac{1}{\beta^4} \sum_{l=1}^{\infty} c_l^{(4)} \frac{\cot \lambda_l}{\lambda_l}\right],$$

because of expressions (1.95) and (1.98).

Expression (1.115) is found in a similar fashion.

The average value of σ_z is given by

$$\left(\frac{\overline{\sigma_z}}{T}\right)_{r=1} = \cos 2\varphi \frac{1}{\beta^2}\left[-\frac{12\nu}{1+\nu} \operatorname{Re} \sum_{l=1}^{\infty} c_l^{(4)} \frac{1}{\lambda^2_l}\right]. \tag{1.117}$$

It should be additionally noted that for a very *thick* plate ($\beta \to 0$), the values of $\frac{\overline{\sigma}_z}{T}$ and $\frac{\overline{\sigma}_\phi}{T}$ are known from the theory of the plane state of deformation. By means of formulas (1.109) and (1.112) it is thus possible to compute the magnitudes D and $\beta^2 \sum_{l=1}^{\infty} \frac{d_l}{\lambda^2_l}$ for $\beta = 0$.

$$\left.\begin{aligned} \operatorname{Re} \sum_{l=1}^{\infty} \frac{d_l}{\lambda^2_l} &= -\frac{2\nu}{\beta^2} \\ D &= \frac{1-\nu}{2} \end{aligned}\right] \quad \beta \to 0. \tag{1.118}$$

is thus found.

APPENDIX /34

Completeness of the Potential Function (A, B_1, B_2) System Used

It is not certain in advance that the potential function (A, B_1, B_2)

system used is complete, i.e., that with every solution (u, v, w) of expression (1.1), potentials A, B_1, B_2 can be found such that the functions u, v, and w of the (A, B_1, B_2) system can be derived by combining expressions (1.4), (1.6), and (1.8).

When an *arbitrary* three-dimensional problem is treated, it is in general certain that the solution can be described by *four* independent potential functions (Ref. 22), while for a large class of problems it has also been established that description by means of *three* potential functions is possible (Ref. 23). It may be expected on physical grounds that, even though there is no mathematical proof for this, "in general" even problems which fall outside the above-mentioned class can be described with only three potential functions. For an extensive class of problems, which also includes the problem formulated in Chapter I, it is, however, possible to give a mathematical proof of the completeness of the potential function (A, B_1, B_2) system employed. This class of problems embraces the problems which pertain to determining displacements in bodies which are convex in one direction. (A body is called convex in one direction if every rectilinear element parallel to the direction connecting two points in the body lies entirely in that body.) If one or more of the dimensions of the body become infinitely great, the displacements at infinity satisfy certain conditions. In proving the assumption that the potential function (A, B_1, B_2) system is complete, a central role is played by the following lemma concerning harmonic functions -- i.e., functions which satisfy the Laplace equation.

Lemma /35

If p(x, y, z) is a harmonic function in region D, it is possible to find another harmonic function q(x, y, z) defined by

$$\frac{\partial q}{\partial z} = p, \tag{A.1}$$

if one of the following conditions is fulfilled:

(a) Region D is convex in the z-direction.

(b) If $r = \sqrt{x^2 + y^2} \to \infty$, then $\left(\frac{\partial p}{\partial z}\right)_{z=0} = 0\,(r^{-3})$.

The proof of the lemma takes a course which is entirely analogous to the proof that is given by Eubanks and Sternberg (Ref. 23) for bodies convex in the z-direction and finite in every plane parallel to the x-y-plane. From the given harmonic function, p(x, y, z) is first derived by means of

$$q_1 = \int_0^z p(x, y, \zeta)\, d\zeta. \tag{A.2}$$

Function q_1 satisfies

$$\frac{\partial q_1}{\partial z} = p,$$

and is thus a solution of

$$\frac{\partial}{\partial z} \Delta q_1 = 0.$$

From this it follows that Δq_1 is independent of z; this is expressed by

$$\Delta q_1 = h(x, y). \tag{A.3}$$

From expression (A.3) it follows by direct calculation that /36

$$\left.\begin{aligned} h(x, y) = \Delta q_1 = \frac{\partial p}{\partial z} + \int_0^z \left(\frac{\partial^2 p}{\partial x^2} + \right. \\ \left. + \frac{\partial^2 p}{\partial y^2} \right) d\zeta = \left(\frac{\partial p}{\partial z} \right)_{z=0} . \end{aligned}\right] \tag{A.4}$$

From condition b it may be concluded that h(x, y) may be integrated over the whole plane z = 0. To function q_1 a function $q_2(x, y)$ is added which is defined by

$$q_2 = -\frac{1}{2\pi} \iint_B h(\xi; \eta) \log \sqrt{(x-\xi)^2 + (y-\eta)^2}\, d\xi\, d\eta, \tag{A.5}$$

which (Ref. 30) obviously fulfills

$$\Delta q_2 = -h(x, y). \tag{A.6}$$

Region B over which integral (A.5) is calculated is the intersection of plane z = 0 with the body. It is now clear that the requred function q(x, y, z) is given by

$$q = q_1 + q_2. \tag{A.7}$$

Note. The hypothesis may be extended to higher derivatives. If p(x, y,z) is a harmonic function, a function q(x, y, z) may be found which fulfills

$$\frac{\partial^n q}{\partial z^n} = p; \ \Delta q = 0, \tag{A.8}$$

on condition a, while condition b must be reinforced to

$$\left(\frac{\partial p}{\partial z} \right)_{z=0} = 0(r^{-2n-1}).$$

The completeness theorem is now formulated as follows.

Theorem. In a body which is convex in the z-direction, a harmonic potential function (A, B_1, B_2) system can be found for every solution (u, v, w) of expression (1.1) such that when $\alpha - \gamma = 3 - 4\nu$ /37

$$\left.\begin{aligned} u &= \frac{\partial A}{\partial y} + \frac{\partial B_1}{\partial x} + \alpha \frac{\partial B_2}{\partial x} + z \frac{\partial^2 B_2}{\partial x \partial z}, \\ v &= -\frac{\partial A}{\partial x} + \frac{\partial B_1}{\partial y} + \alpha \frac{\partial B_2}{\partial y} + z \frac{\partial^2 B_2}{\partial y \partial z}, \\ w &= \frac{\partial B_1}{\partial z} + \gamma \frac{\partial B_2}{\partial z} + z \frac{\partial^2 B_2}{\partial z^2}, \end{aligned}\right] \tag{A.9}$$

and when $r = \sqrt{x^2 + y^2} \to \infty$, the following is true

$$u, v = 0(r^{-1-\varepsilon}) \text{ when } \varepsilon > 0,$$

$$\frac{\partial w}{\partial z} = 0(r^{-3}),$$

$$\frac{\partial \varepsilon_v}{\partial z} = 0(r^{-3}).$$

The proof of this theorem will be provided by solving equations (A.9) for the harmonic functions A, B_1, B_2 with an arbitrarily given solution u, v, w of the elasticity equations (1.1). First of all, B_2 is determined from the relationship

$$\varepsilon_v = -2(1-2\nu)\frac{\partial^2 B_2}{\partial z^2}. \tag{A.10}$$

obtained by differentiating the first equations (A.9) with respect to x, the second equation with respect to y, and the third equation with respect to z, and adding. Because ε_v is harmonic, B_2 may be computed from the lemma.

Next, harmonic function B_1 is calculated from the third equation (A.9). From the third equation (1.1) and from (A.10) it follows that /38

$$\Delta\left(w - \gamma\frac{\partial B_2}{\partial z} - z\frac{\partial^2 B_2}{\partial z^2}\right) = 0, \tag{A.11}$$

so that B_1 again may be computed from the lemma.

To complete the proof, new functions Q and R defined by

$$\left.\begin{aligned} Q &= u - \frac{\partial B_1}{\partial x} - \alpha\frac{\partial B_2}{\partial x} - z\frac{\partial^2 B_2}{\partial x \partial z}, \\ R &= v - \frac{\partial B_1}{\partial y} - \alpha\frac{\partial B_2}{\partial y} - z\frac{\partial^2 B_2}{\partial y \partial z}; \end{aligned}\right] \tag{A.12}$$

are introduced; from equations (1.1) and (A.10) it follows that Q and R are harmonic. By means of the relationship $\alpha - \gamma = 3 - 4\nu$, the third equation (A.9) and (A.10) follows by differentiation of expression (A.12)

$$\frac{\partial Q}{\partial x} + \frac{\partial R}{\partial y} = 0, \tag{A.13}$$

from which it follows that Q and R may be derived from the z-component of a vector potential.

It is now assumed that

$$\left.\begin{aligned} A(x, y, z) = \int_{s_0}^{s} \{-R(\xi, \eta, z)\, d\xi + Q(\xi, \eta, z)\, d\eta\} \\ -\int_0^z d\zeta_1 \int_0^{\zeta_1} d\zeta_2 \left[-\frac{\partial R}{\partial x} + \frac{\partial Q}{\partial y} + \right. \end{aligned}\right] \tag{A.14}$$

$$+\int_{s_0}^{s}\left\{-\frac{\partial^2 R}{\partial \zeta_2^2}\,d\xi+\frac{\partial^2 Q}{\partial \zeta_2^2}\,d\eta\right\}\Bigg].\tag{A.14}$$

In this $\int_{s_0}^{s}$ is defined as the integral along an arbitrary curve which lies entirely in the intersection of the body with the plane z = const. Further, the integrations over ζ_1 and ζ_2 are integrations in the z-direction. It is necessary that point s_o be selected independently of z. In order to show that this is indeed the desired potential function A, we find by differentiation that /39

$$\left.\begin{aligned} Q &= \frac{\partial A}{\partial y}, \\ R &= -\frac{\partial A}{\partial x}, \end{aligned}\right]\tag{A.15}$$

The derivative with respect to z is

$$\left.\begin{aligned} \frac{\partial A}{\partial z} &= \int_0^z d\left(\frac{\partial R}{\partial x}-\frac{\partial Q}{\partial y}\right)+ \\ &+\int_{s_0}^{s}\left(\frac{\partial Q}{\partial z}\right)_{z=0} d\eta-\left(\frac{\partial R}{\partial z}\right)_{z=0} d\xi. \end{aligned}\right]\tag{A.16}$$

From expressions (A.15) and (A.16) it follows that

$$\Delta A = 0,\tag{A.17}$$

so that function A is, in fact, harmonic.

In a simply connected region, this function is also unique. In a multiply connected region, however, A is in general many-valued. Nor is $\frac{\partial A}{\partial z}$ then unique, but $\frac{\partial A}{\partial x}$, $\frac{\partial A}{\partial y}$ and $\frac{\partial^2 A}{\partial z^2}$ may well be.

It should be noted that in the proof the value $\nu = \frac{1}{4}$ offers no particular difficulty, as is the case if the representation with four potentials according to Papkovich is reduced to a representation with three components of the vector potential (Ref. 23). The proof given here is valid for all values of ν in the region $-1 < \nu < \frac{1}{2}$.

The theorem as formulated on page 30 cannot be directly applied to the problem of Chapter I because the solution of this problem does not fulfill the conditions which are set in the theorem regarding the behavior when $r \to \infty$. By splitting off the elementary part from stress distribution (1.33), we obtain a problem, however, to which the completeness theorem which is proven here may be well adapted.

CHAPTER II

NUMERICAL RESULTS FOR THE PLATE LOADED IN ITS PLANE

2.1 The Numerical Calculations

In order that conclusions may be reached in regard to the stresses which are sought, the equations must be solved numerically. The computations consist of:

(a) determining the eigen values;
(b) determining the necessary values of the K-functions;
(c) calculating the required coefficients;
(d) solving the algebraic equations;
(e) computing the stress magnitudes.

In the formulas, the magnitude ν, Poisson's number, occurs. This number plays an important role since, for $\nu = 0$, the problem is elementary, and complexity is thus merely the result of ν not being equal to zero. In the computations, ν is always taken to equal $\frac{1}{4}$, the average of the extreme values which are theoretically possible; the value adopted is furthermore a good approximation of Poisson's number for steel.

(a) *The eigen values are calculated* for equation (1.53) by replacing this equation by two real equations

$$\left.\begin{aligned} \sin 2\mu_l \cosh 2\nu_l &= -2\mu_l \\ \cos 2\mu_l \sinh 2\nu_l &= -2\nu_l. \end{aligned}\right] \qquad (2.1)$$

after setting $\lambda_\ell = \mu_\ell + i\nu_\ell$, ($\mu_\ell$, ν_ℓ real). The trivial (useless) solution $\mu_\ell = \nu_\ell = 0$ is disregarded. We restrict ourselves to the solutions for which $\mu_\ell > 0$ and $\nu_\ell > 0$. From expression (2.1) it immediately follows that /41 $\sin 2\mu_\ell$ and $\cos 2\mu_\ell$ must simultaneously be negative, from which it becomes clear that $2\mu_\ell$ lies in the third quadrant and thus may be written

$$2\mu_l = (2l - \frac{1}{2})\pi - \varepsilon_l, \quad (l = 1, 2 \ldots) \qquad (2.2)$$

in which ε_ℓ is to be determined more exactly.

An arbitrary (small) initial value us given to ε_ℓ. From the first equation (2.1), we calculate a value pertaining to ν_ℓ. Then a corrected value ε_ℓ is determined from the second equation (2.1). This iteration process quickly converges. Table I gives a survey of the solutions for $\ell = 1, \ldots 7.$*

* According to the computation of the eigen values by means of Hayashi's tables of hyperbolic and goniometric functions (Ref. 26), it was clear that these eigen values had already been determined by other authors (Refs. 27, 28, 29). The results obtained here agree with the literature in the figures quoted.

Because of the properties of the hyperbolic functions, it is also simple to posit an asymptotic formula for λ_ℓ. For large values of ℓ it is obvious that

$$\left.\begin{aligned} \lambda_l \rightarrow \left(l - \frac{1}{4}\right)\pi - \frac{1}{(4l-1)\pi}\log(4l-1)\pi + \\ + \frac{1}{2}\, i \log(4l-1)\pi + \ldots . \end{aligned}\right] \tag{2.3}$$

For a value of ℓ as low as 5, this formula gives 14.8541 + i 2.045, in satisfactory agreement with the more accurate value.

(b) *The K-value computation* extends not only over the real, but also over the complex magnitudes. $K(k\pi\beta)$ and $K(\lambda_\ell\beta)$ occur in the formulas. With sufficiently large values of β the asymptotic formulas, which here quickly lead to a result, are always adequate. For small values of k, ℓ, and β, tables were at our disposal which, although they were intended for calculating Bessel func- /42
tions with complex arguments, here, however, prove to be useful only for calculating functions with real arguments. The functions with complex arguments are determined by means of series expansions. The formulas used are once more summarized here. The asymptotic formula is

$$\frac{K_1(z)}{K_2(z)} = \frac{1 + \frac{3}{8} z^{-1} - \frac{15}{128} z^{-2} + \ldots}{1 + \frac{15}{8} z^{-1} + \frac{105}{128} z^{-2} + \ldots}, \tag{2.4}$$

while the series expansions for $K_1(z)$ and $K_2(z)$ are

$$\left.\begin{aligned} K_1(z) = -\frac{1}{z} - \sum_{r=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{1+2r}}{r!\,(1+r)!} \Big\{ \log\left(\frac{z}{2}\right) + \\ + \gamma - \frac{1}{2}\sum_{m=1}^{r+1}\frac{1}{m} - \frac{1}{2}\sum_{m=1}^{r}\frac{1}{m} \Big\}, \end{aligned}\right] \tag{2.5}$$

$$\left.\begin{aligned} K_2(z) = \frac{2}{z^2} - \frac{1}{2} - \sum_{r=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2+2r}}{r!\,(2+r)!} \Big\{ \log\left(\frac{z}{2}\right) + \\ + \gamma - \frac{1}{2}\sum_{m=1}^{r+2}\frac{1}{m} - \frac{1}{2}\sum_{m=1}^{r}\frac{1}{m} \Big\}, \end{aligned}\right] \tag{2.6}$$

in which γ represents Euler's constant (γ = 0.5772157...). The abbreviation

$$K^*(z) = \frac{K_1(z)}{K_2(z)} \tag{2.7}$$

is introduced for calculating the K values, from which K(z) is determined by

$$K(z) = zK^*(z) - 2. \tag{2.8}$$

TABLE I /43

EIGEN VALUES AND K*-VALUES

k of l	$k\pi$	λ_l	$K^*\left(\frac{k\pi}{2}\right)$	$K^*(k\pi)$	$K^*(\frac{1}{2}\lambda_l)$	$K^*(\lambda_l)$	$K^*(\frac{3}{2}\lambda_l)$	$K^*(2\lambda_l)$	$K^*(3\lambda_l)$
1	3,1416	2,1062+i 1,1254	—0,4875	—0,6637	—0,4012—i 0,1185	—0,6273—i 0,1280	—0,7055—i 0,1103	—0,7654—i 0,0964	—0,8347—i 0,0736
2	6,2832	5,3564+i 1,552	—0,6637	—0,8019	—0,6354—i 0,0640	—0,7859—i 0,0496	—0,8484—i 0,0378	—0,8825—i 0,0305	—0,9192—i 0,0217
3	9,4248	8,5367+i 1,776	—0,7504	—0,8599	—0,7373—i 0,0414	—0,8520—i 0,0267	—0,8971—i 0,0194	—0,9211—i 0,0153	—0,9463—i 0,0107
4	12,5664	11,6992+i 1,929	—0,8019	—0,8916	—0,7937—i 0,0277	—0,8869—i 0,0168	—0,9221—i 0,0120	—0,9406—i 0,0093	—0,9598—i 0,0064
5	15,7080	14,8541+i 2,047	—0,8358	—0,9116	—0,8301—i 0,0199	—0,9084—i 0,0166	—0,9373—i 0,0081	—0,9524—i 0,0063	—0,9678—i 0,0043
6	18,8496	18,0049+i 2,140	—0,8599	—	—0,8557—i 0,0150	—	—	—	—
7	21,9911	21,1534+i 2,220	—0,8778	—	—0,8745—i 0,0117	—	—	—	—

TABLE IIa /44

VALUES OF $\chi_{\ell k}^{*(1)}$ FOR $\beta = 1/2$

k \ l	1	2	3	4	5	6	7
1	—0,08477+i 0,16729	0,00989—i 0,03952	0,00546—i 0,00506	0,00221—i 0,00135	0,00109—i 0,00051	0,00061—i 0,00024	0,00038—i 0,00013
2	—0,01101+i 0,02008	—0,03102+i 0,03574	0,00506—i 0,01978	0,00373—i 0,00324	0,00172—i 0,00097	0,00092—i 0,00040	0,00008—i 0,00020
3	—0,00604+i 0,00804	—0,00004+i 0,00440	—0,02092+i 0,01995	0,00325—i 0,01418	0,00297—i 0,00273	0,00147—i 0,00081	0,00082—i 0,00035
4	—0,00372+i 0,00441	—0,00217+i 0,00865	0,00119+i 0,00349	—0,01658+i 0,01424	0,00194—i 0,01160	0,00254—i 0,00227	0,00131—i 0,00072
5	—0,00251+i 0,00279	—0,00080+i 0,00057	0,00014+i 0,00079	0,00141+i 0,00325	—0,01419+i 0,01134	0,00131—i 0,01010	0,00226—i 0,00208
6	—0,00180+i 0,00193	—0,00067+i 0,00035	—0,00017+i 0,00032	0,00045+i 0,00072	0,00143+i 0,00312	—0,01265+i 0,00960	0,00087—i 0,00909
7	—0,00136+i 0,00145	—0,00054+i 0,00024	—0,00023+i 0,00017	0,00007+i 0,00027	0,00059+i 0,00071	0,00138+i 0,00304	—0,01156+i 0,00842

TABLE IIb /45

VALUES OF $\chi^{*(2)}_{\ell k}$ FOR $\beta = 1/2$

k \ l	1	2	3
1	—0,07396+i 0,20822	0,01022—i 0,04566	0,00599—i 0,00571
2	—0,00124+i 0,02385	—0,04458+i 0,07407	0,00630—i 0,03211
3	—0,00148+i 0,00909	0,00877+i 0,01097	—0,04475+i 0,05801
4	—0,00108+i 0,00488	0,00117+i 0,01729	0,01016+i 0,01235
5	—0,00078+i 0,00306	0,00197+i 0,00136	0,00510+i 0,00324
6	—0,00058+i 0,00210	0,00120+i 0,00078	0,00286+i 0,00136
7	—0,00047+i 0,00167	0,00080+i 0,00051	0,00182+i 0,00073

k \ l	4	5	6	7
1	0,00241—i 0,00149	0,00118—i 0,00056	0,00066—i 0,00026	0,00040—i 0,00013
2	0,00545—i 0,00502	0,00246—i 0,00145	0,00129—i 0,00058	0,00076—i 0,00028
3	0,00499—i 0,03139	0,00580—i 0,00548	0,00279—i 0,00164	0,00152 i 0,00068
4	—0,04770+i 0,05332	0,00319—i 0,03334	0,00639—i 0,00626	0,00321 i 0,00190
5	0,01081+i 0,01425	—0,05144+i 0,05185	0,00214—i 0,03602	0,00705 i 0,007[illegible]
6	0,00590+i 0,00377	0,01137+i 0,01622	0,05548+i 0,05187	0,0[illegible]110 i 0,03[illegible]
7	0,00341+i 0,00154	0,00658+i 0,00434	0,01183+i 0,01821	[illegible] i 0,05244

A number of $K^*(z)$ values are collected in Table I. /50

(c) *Coefficients* $\chi^{*(n)}_{\ell k}$ *are calculated in various stages* by means of the K-values given in Table I.

This is done for β-values of $^1/_2$, 1, $^3/_2$, 2 and 3.

The results of these calculations are given in Tables IIa through IIj.

(d) *System (1.87) and system (1.99)*, when $\beta = 1$, $^3/_2$, 2 and 3 *are reduced* to ten real equations with ten unknowns, while when $\beta = {}^1/_2$ an expansion into 14 equations with 14 unknowns proved to be necessary. The effect of this is discussed in Section 2.2.

Table III contains the solutions obtained. C and D are also tabulated (computed from expression [1.73]) and the magnitude

$$F = \frac{1}{\frac{\pi^2(1+\nu)}{24\,\nu}\beta^2 + \frac{\nu\beta}{2}\quad \mathrm{Re}\sum_{l=1}^{\infty}\frac{c_l}{\lambda^3_l}\,K(\lambda_l\,\beta)}\,. \tag{2.9}$$

When judging the accuracy of the calculations it should be kept in mind that here it is a question of values for correcting the classical stress magnitudes which are rarely greater than 10% of the uncorrected. Accuracy to three, or at least to two, digits is therefore considered satisfactory.

TABLE IIc

VALUES OF $\chi_{\ell k}^{*(1)}$ FOR $\beta = 1$

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k \ l	1	2	3	4	5
1	—0,16281+i 0,28443	0,02792—i 0,08777	0,01359—i 0,01191	0,00562—i 0,00330	0,00281—i 0,00129
2	—0,01924+i 0,02853	—0,08411+i 0,08896	0,01566—i 0,05486	0,01097—i 0,00918	0,00512—i 0,00281
3	—0,01155+i 0,01031	0,00013+i 0,01168	—0,06323+i 0,05736	0,01080—i 0,04338	0,00944—i 0,00805
4	—0,00740+i 0,00540	—0,00204+i 0,00293	0,00346+i 0,01071	—0,05350+i 0,04424	0,00684—i 0,03770
5	—0,00509+i 0,00334	—0,00211+i 0,00128	0,00041+i 0,00241	0,00435+i 0,01056	—0,04759+i 0,03693

TABLE IId

VALUES OF $\chi_{\ell k}^{*(2)}$ FOR $\beta = 1$

k \ l	1	2	3	4	5
1	—0,12361+i 0,53392	0,03457—i 0,13427	0,01884—i 0,01732	0,00767—i 0,00465	0,00378—i 0,00178
2	0,03405+i 0,05532	—0,19452+i 0,34709	0,02941—i 0,14782	0,02544—i 0,02321	0,01153—i 0,00674
3	0,01306+i 0,01855	0,05605+i 0,05879	—0,24775+i 0,32771	0,02792—i 0,17569	0,03254—i 0,03071
4	0,00667+i 0,00940	0,02711+i 0,01644	0,06480+i 0,07595	—0,29409+i 0,33064	0,01982—i 0,20611
5	0,00401+i 0,00572	0,01536+i 0,00736	0,03448+i 0,02076	0,07147+i 0,09318	—0,33659+i 0,33977

TABLE IIe

VALUES OF $\chi_{\ell k}^{*(1)}$ FOR $\beta = {}^{3}/_{2}$

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k \ l	1	2	3	4	5
1	—0,2777+i 0,4312	0,0536—i 0,1580	0,0255—i 0,0219	0,0106—i 0,0062	0,0053—i 0,0024
2	—0,0345+i 0,0403	—0,1638+i 0,1689	0,0323—i 0,1083	0,0222—i 0,0183	0,0109—i 0,0057
3	—0,0209+i 0,0136	—0,0001+i 0,0224	—0,1294+i 0,1145	0,0229—i 0,0890	0,0197—i 0,0166
4	—0,0134+i 0,0069	—0,0042+i 0,0055	0,0067+i 0,0219	—0,1119+i 0,0910	0,0147—i 0,0790
5	—0,0093+i 0,0041	—0,0043+i 0,0023	0,0007+i 0,0050	0,0088+i 0,0221	—0,1008+i 0,0774

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TABLE IIf

VALUES OF $\chi_{\ell k}^{*(2)}$ FOR $\beta = {}^{3}/_{2}$

k \ l	1	2	3	4	5
1	—0,2078+i 1,1804	0,0829—i 0,3140	0,0448—i 0,0408	0,0183—i 0,0110	0,0090—i 0,0042
2	0,1144+i 0,1260	—0,5451+i 0,9849	0,0838—i 0,4189	0,0724—i 0,0658	0,0328—i 0,0192
3	0,0477+i 0,0408	0,1717+i 0,1762	—0,7482+i 0,9905	0,0849—i 0,5314	0,0988—i 0,0930
4	0,0255+i 0,0203	0,0856+i 0,0502	0,2021+i 0,2363	—0,9168+i 1,0298	0,0620—i 0,6428
5	0,0159+i 0,0122	0,0494+i 0,0227	0,1094+i 0,0656	0,2264+i 0,2953	—1,0678+i 1,0773

TABLE IIg

VALUES OF $\chi_{\ell k}^{*(1)}$ FOR $\beta = 2$

k \ l	1	2	3	4	5
1	—0,4381+i 0,6255	0,0881—i 0,2498	0,0412—i 0,0350	0,0172—i 0,0099	0,0087—i 0,0039
2	—0,0560+i 0,0566	—0,2722+i 0,2716	0,0549—i 0,1803	0,0373—i 0,0306	0,0176—i 0,0095
3	—0,0337+i 0,0182	—0,0009+i 0,0371	—0,2195+i 0,1912	0,0394—i 0,1511	0,0338—i 0,0283
4	—0,0217+i 0,0089	—0,0072+i 0,0090	0,0109+i 0,0371	—0,1918+i 0,1544	0,0258—i 0,1356
5	—0,0150+i 0,0052	—0,0074+i 0,0037	0,0010+i 0,0084	—0,0112—i 0,0392	—0,1742+i 0,1324

TABLE IIh

VALUES OF $\chi_{\ell k}^{*(2)}$ FOR $\beta = 2$

k \ l	1	2	3	4	5
1	—0,3598+i 2,2963	0,1655—i 0,6208	0,0894—i 0,0801	0,0364—i 0,0219	0,0180—i 0,0084
2	0,2601+i 0,2568	—1,1862+i 2,1466	0,1837—i 0,9140	0,1583—i 0,1438	0,0719—i 0,0418
3	0,1117+i 0,0830	0,3865+i 0,3950	—1,6830+i 2,2257	0,1917—i 1,1958	0,2229—i 0,2091
4	0,0608+i 0,0410	0,1952+i 0,1139	0,4604+i 0,5387	—2,0926+i 2,3480	0,1438—i 1,4676
5	0,0381+i 0,0246	0,1135+i 0,0518	0,2511+i 0,1507	0,5201+i 0,6791	—2,4604+i 2,4749

TABLE IIi

VALUES OF $\chi^{*(1)}_{\ell k}$ FOR $\beta = 3$

k \ l	1	2	3	4	5
1	—0,8810+i 1,1437	0,1828—i 0,4996	0,0843—i 0,0709	0,0354—i 0,0202	0,0179—i 0,0080
2	—0,1170+i 0,1013	—0,5735+i 0,5548	0,1183—i 0,3797	0,0796—i 0,0648	0,0375—i 0,0201
3	—0,0699+i 0,0306	—0,0039+i 0,0777	—0,4712+i 0,4037	0,0861—i 0,3248	0,0731—i 0,0609
4	—0,0450+i 0,0143	—0,0162+i 0,0187	0,0224+i 0,0797	—0,4162+i 0,3316	0,0570—i 0,2944
5	—0,0311+i 0,0082	—0,0161+i 0,0077	0,0017+i 0,0181	0,0316+i 0,0823	—0,3806+i 0,2872

TABLE IIj

VALUES OF $\chi^{*(2)}_{\ell k}$ FOR $\beta = 3$

k \ l	1	2	3	4	5
1	—0,9238+i 6,4406	0,4740—i 1,7621	0,2542—i 0,2303	0,1040—i 0,0624	0,0516—i 0,0240
2	0,8259+i 0,7665	—3,6978+i 6,6822	0,5763—i 2,8526	0,4953—i 0,4492	0,0111—i 0,0707
3	0,3642+i 0,2499	1,2372+i 1,2644	—5,4045+i 7,1338	0,6184—i 3,8405	0,7170—i 0,6720
4	0,2008+i 0,1240	0,6330+i 0,3692	1,4939+i 1,7507	—6,8106+i 7,6313	0,4707 i 4,7770
5	0,1268+i 0,0745	0,3708+i 0,1689	0,8210+i 0,4937	1,7017+i 2,2252	—8,0697+i 8,1087

The accuracy attained is determined by the following factors: Accuracy to which eigen values λ_1 are known, the rounding off of the digits, and the systematic error from reducing the system of equations.

The imaginary parts of the eigen values are calculated to four places, which is considered sufficient for the result to be attained. The corresponding real parts of the eigen values are accurate to four figures after the decimal point. Rounding off of figures, in general, plays no role since many decimals are always used.

The most important error comes from reducing the system of equations. For thin plates the limitation to 10 equations suffices; for the case $\beta = 1/2$ it was necessary to use 14 equations. For practical reasons this number is considered the limit, and the accuracy achieved seems, on the basis of the re- /53 sults obtained for the stress distribution, to be adequate (see Section 2.2). A first impression of the accuracy attained has already been obtained by comparing the unknowns $c_1, \ldots, c_5$ calculated from ten equations with the values of these unknowns computed from 14 equations (Table IV).

As it appears, the stability of the first c_ℓ coefficients is reasonably good. The small changes have little effect on the stress distribution. Of much greater significance, however, is the reduction itself. It is then also clear that the effect of the sixth and seventh term on the stress values is by no means to be neglected. From the values of $c_1, \ldots, c_7$ it may, however, be hoped that neglecting $c_8, \ldots$ will bring no more great changes in the results obtained. This will be dealt with in more detail when the results are discussed (Section 2.2).

It would be illusory to expect that mistakes in computation could be entirely excluded in the extremely voluminous computations. The attempt is indeed always made, so far as possible, to include control calculations to reduce the danger of computational errors to a minimum. As an example of such a control calculation, we would like to mention relationship (1.115) which is then always used. The graphic presentation of results of intermediate computations provides a check on the final results, which is effective in many respects.

System (1.99) for the first approximation of the asymptotic solution for the thin plate is reduced to 10 equations. The solution is summarized in Table V. The accuracy achieved here is great, as is also clear from comparison of these results with the results for computations involving only two equations, which are also given in Table V.

2.2 Discussion of Results

The principal results of the computations are shown in Figures 1 through 10.

In Figure 1 the magnitude D, the coefficient of plane state of stress (1.31), is calculated as a function of ratio β of hole diameter to plate

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TABLE III

VALUES OF C, D, F AND c_ℓ

β	C	D	F	c_1	c_2	c_3
½	0,5066	0,3916	1,9045	—8,1681—i 2,3872	—3,4001—i 1,1005	—1,5193—i 1,1896
1	0,3144	0,3971	0,4828	—5,0825—i 0,9747	—1,2626—i 0,6502	—0,3279—i 0,5620
3/2	0,2793	0,3993	0,2155	—3,3231—i 0,4313	—0,4564—i 0,4824	—0,0286—i 0,3317
2	0,2667	0,3997	0,1214	—2,2459—i 0,1907	—0,1924—i 0,3342	0,0752—i 0,2029
3	0,2574	0,3999	0,0540	—1,2067—i 0,0410	—0,0589—i 0,1927	0,0457—i 0,1128

β	c_4	c_5	c_6	c_7
½	1,3894—i 1,3966	—1,2512+i 0,9249	—0,6953—i 0,3103	—0,0758—i 0,3824
1	—0,0056—i 0,4023	0,1406—i 0,2334	—	—
3/2	0,0786—i 0,2108	0,1111—i 0,1104	—	—
2	0,1449—i 0,0543	—	—	—
3	0,0520—i 0,0609	—	—	—

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TABLE IV

COMPARISON OF THE CALCULATED VALUES OF c_1 THROUGH c_5 FROM 10 OR 14 EQUATIONS

	c_1	c_2	c_3	c_4	c_5
10 equa.	—8,1508—i 2,3758	—3,3267—i 1,0910	—1,3710—i 1,1867	1,5846—i 1,3807	—0,8995+i 0,9359
14 equa.	—8,1681—i 2,3872	—3,4001—i 1,1001	—1,5193—i 1,1896	1,3894—i 1,3966	—1,2512+i 0,9249

TABLE V

VALUES OF COEFFICIENTS $c_\ell^{(1)}$ OF THE ASYMPTOTIC EXPANSION

	$c_1^{(4)}$	$c_2^{(4)}$	$c_3^{(4)}$	$c_4^{(4)}$	$c_5^{(4)}$
2 equa.	1,4506—i 0,2214	—	—	—	—
10 equa.	1,5229—i 0,1613	—0,0388+i 0,2670	—0,1093+i 0,1321	—0,0862+i 0,0615	—0,0598+i 0,0228

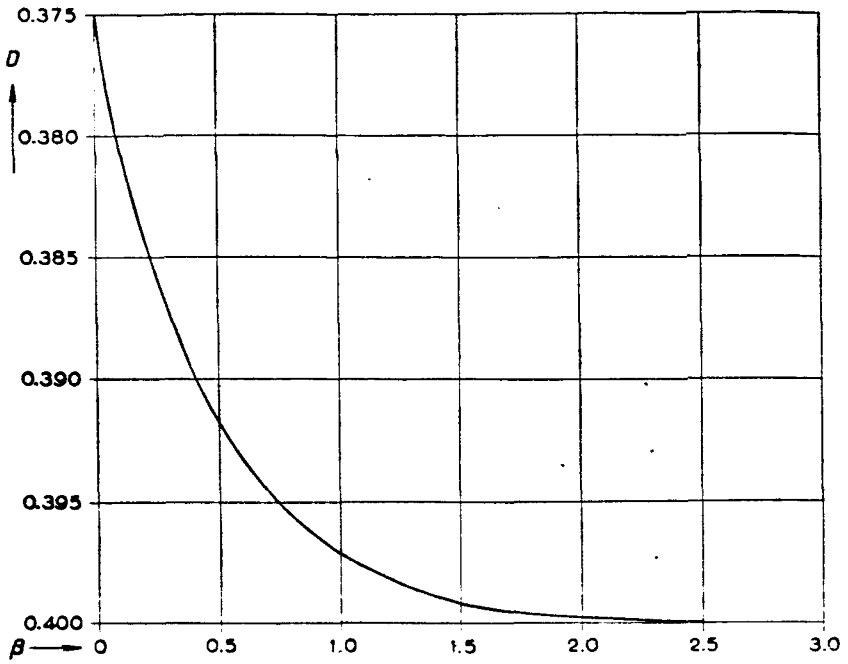

Figure 1

Coefficient D

thickness. Here we make use of the known value of D for $\beta = 0$ given in expression (1.116). It is apparent that it is possible to interpolate the value /54 of D for the $0 < \beta < 0.5$ region with reasonable accuracy.

Figure 2 gives the connection between the *value of normal stress in the direction perpendicular to the plate surface* ($\overline{\sigma}_z$) *averaged* over the plate thickness along the edge of the hole and ratio β. Without any calculating, it can immediately be seen that $\frac{-\overline{\sigma}_z}{T \cos 2\phi} = 2\nu = 0.5$ when $\beta = 0$. From the graph it is clear that for the average value $\overline{\sigma}_z$ there is no need to carry out the extraordinarily cumbersome computation in the $0 < \beta < 0.5$ region. Further, it is apparent that the deviations from the plane state of deformation or the plane state of stress are important in the $0.25 < \beta < 3$ region.

It is interesting to point out here the especially good agreement which exists between the value (-0.167) when $\beta = 1$ and the value found in the literature (-0.169) which Green (Ref. 4) has computed by another method.

Finally, to give an idea of the size of the region in the radial direc- /56 tion where the deviation from the plane state of stress is significant, Figure 3 represents the course of the averaged normal stress $\overline{\sigma}_z$ with radius r for the single value $\beta = 1$. It is clear that at a distance of twice the hole diameter the plane state of stress is approximately unperturbed. It may be expected

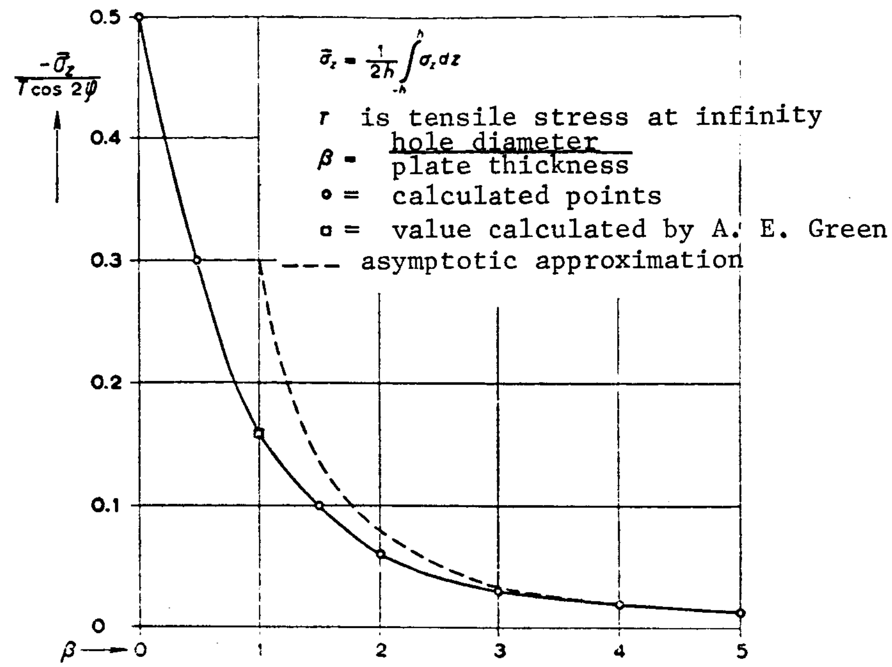

Figure 2 /55

Average Value of σ_z on the Edge of the Hole

that this perturbation penetrates farther in the radial direction when the β values are smaller. The perturbation region is more restricted to the vicinity of median plane z = 0 as the radius becomes larger.

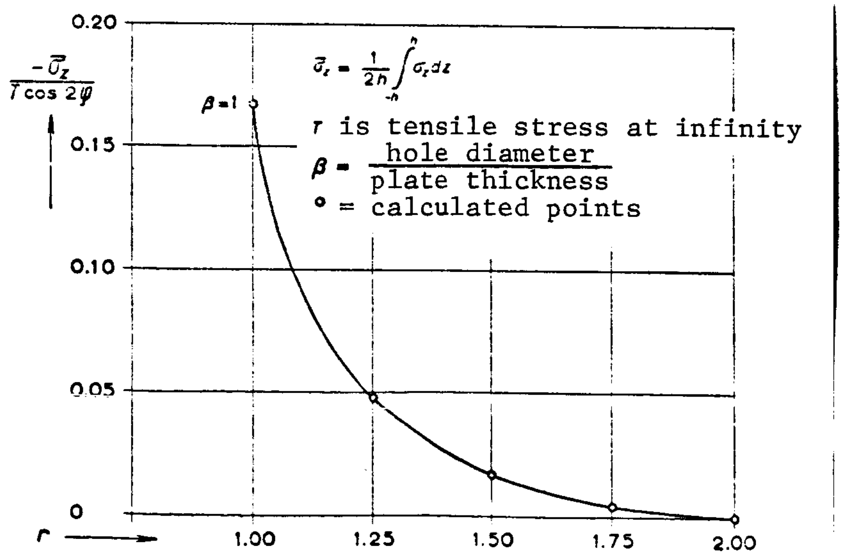

Figure 3

Connection Between $\bar{\sigma}_z$ and Radius r for $\beta = 1$

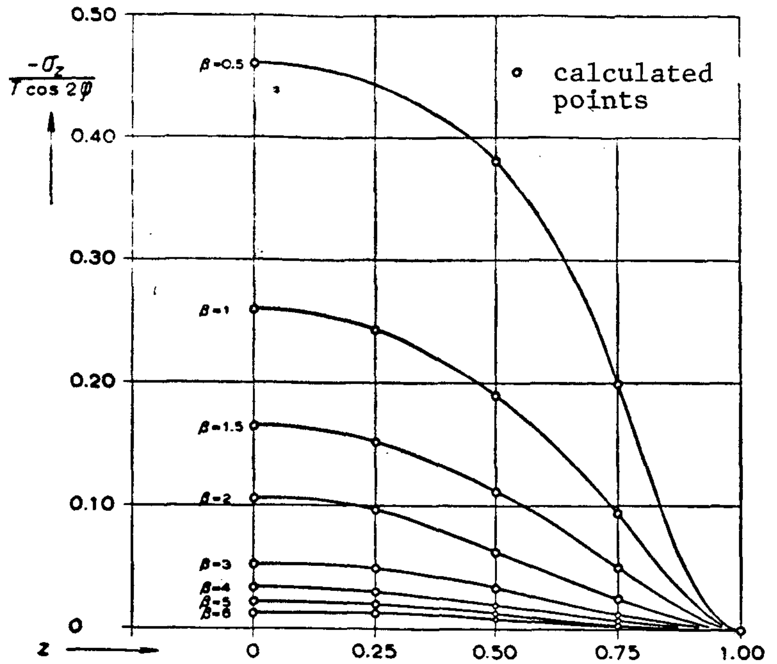

Figure 4

Path of σ_z on the Hole Edge Over the Plate Thickness

Figure 4 gives the *distribution of normal stress* σ_z *on the hole edge over the plate thickness* for various values of β. The most important conclusion that can be drawn from this figure is that in the middle (z = 0), the limiting value 2ν = 0.5 is already almost reached when β = 0.5. The course of σ_z over the plate thickness is also used as a criterion of the accuracy of the numerical calculations; we will return to this when discussing Figure 10.

There is outstanding agreement between the value of $\frac{\sigma_z}{T}$ found here when β = 1 and z = 0, i.e., -0.262, and the value of -0.268 given by Green (Ref. 4). The slight divergence is probably to be attributed to the effect of truncating the series. /57

Figure 5 gives the greatest values of normal stress σ_z, which always occur essentially on median plane z = 0, as a function of $\frac{1}{\beta}$. This figure may be compred with Figure 6 from the work by Sternberg and Sadowsky (Ref. 6). It is evident that the curve calculated here lies closer to the curve given by Sternberg and Sadowsky when ν = 0.20 than to the similar (interpolated) curve when ν = 0.25. This can be simply explained from the lesser accuracy of the approximation method developed by these authors in comparison to the method used here. Qualitatively Sternberg and Sadowsky's results are, however, especially elegant and their method has the advantage that it can be easily adapted to various values of ν.

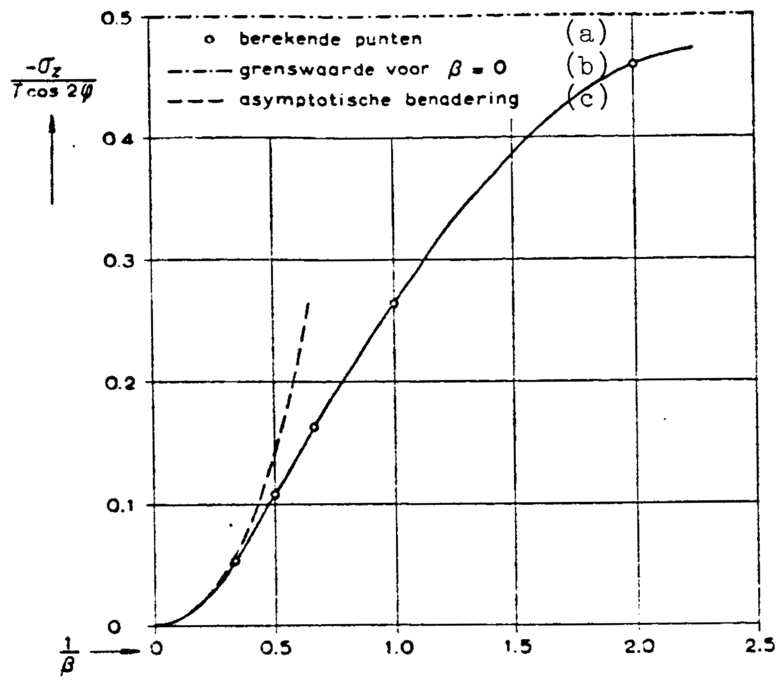

Figure 5 /58

Value of σ_z Along the Hole Edge on the Middle Plane

(a) - Calculated points; (b) - Limiting value when $\beta = 0$; (c) - asymptotic approximation.

The average value of the tangential normal stress σ_ϕ *over the plate thickness* on the hole edge is represented in Figure 6 as a function of β. Here by means of Figures 1 and 2, it is possible to compute the graph down to $\beta = 0$, since

$$\left(\frac{\bar{\sigma}_\varphi}{T}\right)_{r=1} = 1 + \nu\left(\frac{\bar{\sigma}_z}{T}\right)_{r=1} - 4(1+\nu)D\cos 2\varphi. \qquad (2.10)$$

It is evident that the deviations from the value

$$\frac{-\bar{\sigma}_\varphi^{(2)}}{T\cos 2\varphi} = 2,$$

computed either from the theory of plane stress or from that of the plane state of deformation are only slight for all values of β. The maximum of expression (2.10), which is 2.034, lies between 0 and 0.5. The elementary theory obviously suffices for the average value of the normal tangential stress. /59

It should be noted that when $\beta = 1$, the value $\frac{-\bar{\sigma}_\phi^{(2)}}{T\cos 2_\phi} = 2.0276$ found by Green (Ref. 4) is in very good agreement with the value of 2.0270 calculated here.

Figure 7 shows *normal tangential stress* σ_ϕ *along the hole edge* on the

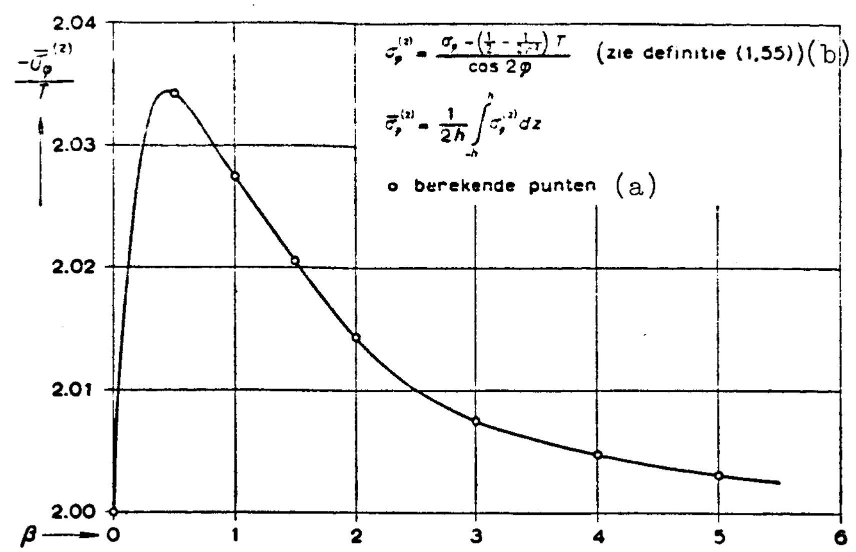

Figure 6

Average Value of $\sigma_\phi^{(2)}$ on the Hole Edge

(a) - Calculated points; (b) - (see definition [1.55]).

external surface as a function of β. When $\beta = {}^1/_2$ this stress value is found to be 2.64. Between $\beta = 0$ and $\beta = {}^1/_2$, the value is extrapolated and represented by a dot-dash line on the figure. More than qualitative significance, however, cannot be attributed to this extrapolation, since larger errors may be expected in the above values than on the above graphs. This is already apparent when we compare the value when $\beta = 1$ with that given by Green, who computes this value from a truncated Fourier series which for $z = 1$ has the form

$$\left.\begin{aligned} \left(\frac{\sigma_\varphi}{T}\right)_{\substack{r=1\\z=1}} &= 1 + \cos 2\varphi \\ &[-2{,}0276 + 0{,}1047 + 0{,}0496 + 0{,}0271 \\ &+ 0{,}0168 + 0{,}0113 + 0{,}0081] = 1 + \cos 2\varphi(-1{,}810), \end{aligned}\right] \tag{2.11}$$

while the value calculated here is found from /60

$$\left.\begin{aligned} \left(\frac{\sigma_\varphi}{T}\right)_{\substack{z=1\\r=1}} &= 1 + \cos 2\varphi\,[-1{,}9854 + 0{,}3017\,\{0{,}9158 \\ &-0{,}0266 - 0{,}0519 - 0{,}0337 - 0{,}0169\}] \\ &= 1 + \cos 2\varphi(-1{,}748). \end{aligned}\right] \tag{2.12}$$

It is obvious that the great discrepancy is for the most part caused by truncation of the series. If it is assumed that series (2.11) and (2.12) are both continued in a regular fashion, an improved estimate can be made. The figures which are thus found are -1.79 and -1.77 for expressions (2.11) and (2.12),

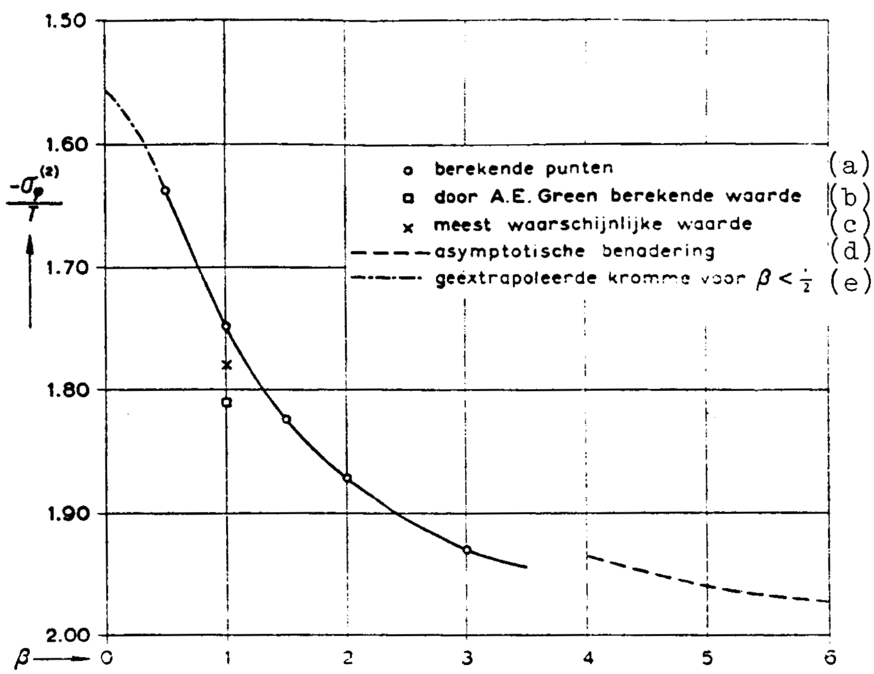

Figure 7

Value of $\sigma_\phi^{(2)}$ Along the Hole Edge on the External Surface

(a) - Computed points; (b) - Value calculated by A. E. Green;
(c) - Most probable value; (d) - Asymptotic approximation;
(e) - Extrapolated curve for $\beta < {}^1/_2$.

respectively. From this it follows that the averaged value (-1.78) of Green's results and those found here may be considered quite reliable.

The effect of truncating the series for $\beta = {}^3/_2$ is smaller, as can be seen from the expansion

$$\left(\frac{\sigma_\varphi}{T}\right)_{\substack{z=1\\r=1}} = 1 + \cos 2\varphi\,[-1{,}9963 + 0{,}3031\,\{0{,}6597 - 0{,}0499 - 0{,}0355 - 0{,}0186 - 0{,}0085\}] = 1 + \cos 2\varphi\,(-1{,}830), \qquad (2.13)$$

which justifies the conclusion that the error in the stress concentration sought will certainly not be greater than 0.01.

It is more difficult to estimate the error in the quoted value when $\beta = {}^1/_2$. The truncated series used to compute this has the form

$$\frac{\sigma_\varphi}{T} = 1 + \cos 2\varphi\,[-1{,}95798 + 0{,}2976\,\{1{,}2291 + 0{,}0344 - 0{,}0858 - 0{,}1435 + 0{,}0746 - 0{,}0113 - 0{,}0163\}] = 1 + \cos 2\varphi\,(-1{,}636). \qquad (2.14)$$

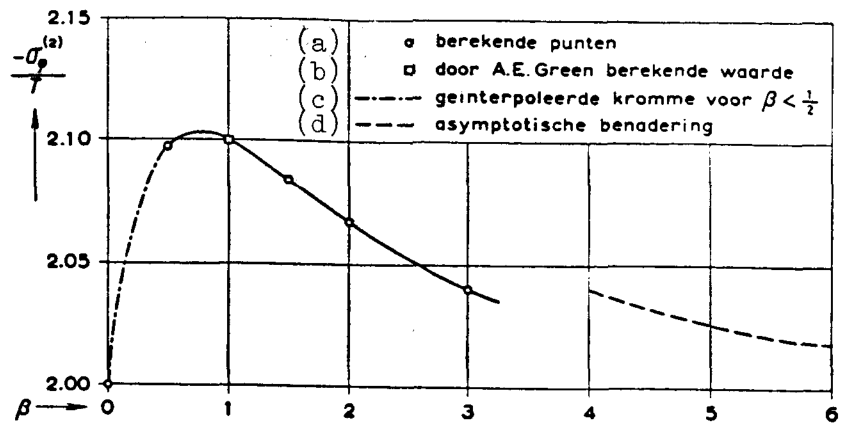

Figure 8

Value of $\sigma_\phi^{(2)}$ Along the Hole Edge on the Median Plane

(a) - Computed points; (b) - Value computed by A. E. Green; (c) - Curve interpolated when $\beta < {}^1/_2$; (d) - Asymptotic approximation.

Normal tangential stress σ_ϕ *along the hole edge in the median plane* is shown in Figure 8. As is to be expected, this normal stress has a maximum which clearly appears for a value of β between 0.5 and 1.0. The value of 2.100 calculated here when $\beta = 1$ is again in excellent agreement with the value of 2.10 given by Green. /61

Figure 9 gives the course of the *normal tangential stress on the hole edge* over the plate thickness. It is clear that the points of intersection of the curve with the straight line $\frac{-\sigma_\phi^{(2)}}{T \cos 2_\phi} = 2$ in elementary theory lie relatively closer to the outside as the value of β increases, i.e., measured in units of the plate thickness. Measured in terms of the hole radius, these intersection points lie closer to the inside with increasing values of β. This outcome is in complete agreement with what may be expected on physical grounds.

Finally, Figure 10 gives some insight into the accuracy achieved with the computations carried out when $\beta = {}^1/_2$. The computed values of normal stress σ_z on the hole edge lie on a smooth curve neither when ten equations are taken into consideration nor when the number is expanded to fourteen. The wave nature of the latter improved solution is noticeably weaker. The noted solid curve in the figure shows the path estimated from the calculations performed. The points calculated by means of 14 equations deviate from this curve by 4% at the most. /62

2.3 Conclusions

A few conclusions may now be drawn on the basis of the computational

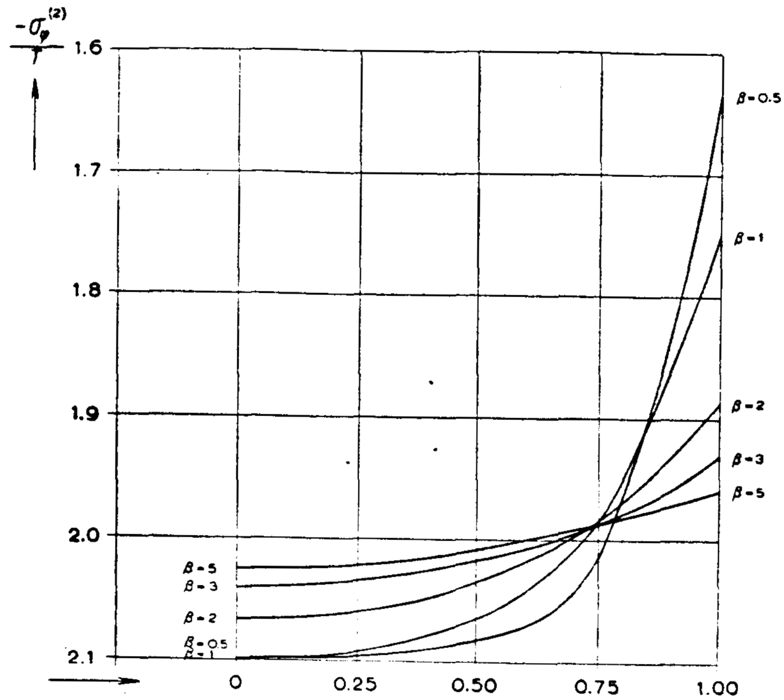

Figure 9

Course of $\sigma_\phi^{(2)}$ Over the Thickness Along the Hole Edge

results.

In the first place it is now possible to determine the accuracy of the theory of the plane state of stress. The computations are, to be sure, restricted to a circular cylindrical hole, but it may be expected that the conclusions drawn from them have a wider scope. It is then clear that the theory of the plane state of stress is in general an excellent approximation. The essence of the situation is that normal stress σ_z deviates greatly perpendicular to the plane of the plate which is assumed to be in plane state of stress zero, but this stress is relatively small. The most important stress, that is, the greatest value of normal tangential stress, is excellently approximated by the theory of plane stress: The stress concentration factor in the middle of the plate (where the greatest stress occurs) is, with the most unfavorable hole-diameter/plate-thickness ratio, no more than 3% greater than the stress concentration factor 3 of the two-dimensional theory. For this reason the stress concentration factors which are computed by means of the two-dimensional theory may be considered reliable for plates loaded in their plane. /63

On the exterior planes of the plate the deviation of the stress distribution from the one according to two-dimensional theory is larger. The stress concentration factor can fall 10 to 15% below its classical value in this case. This outcome is particulary of significance for determining residual stresses with the Mathar-Soete method. We will return to this in /64

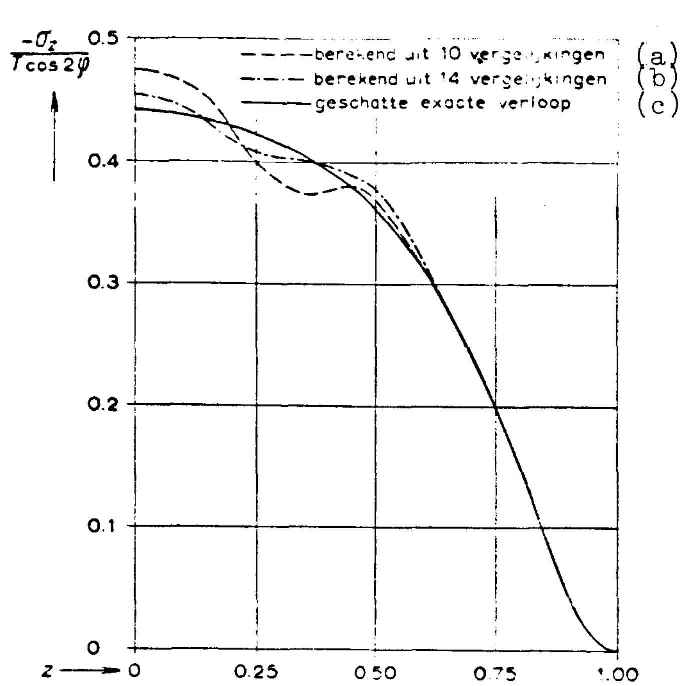

Figure 10

The Value of σ_z on the Hole Edge for $\beta = 1/2$

(a) - Computed from 10 equations; (b) - Computed from 14 equations; (c) - Estimated exact path.

Chapter V. It suffices here to remark that this deviation from the two-dimensional theory quickly disappears with increasing distance from the hole edge.

CHAPTER III

THEORY OF A PLATE LOADED IN BENDING /65

3.1 Introduction

This chapter will deal with the bending of an infinitely extended isotropic plate with a cylindrical perforation. While in Chapter I, only those solutions were noted which were symmetrical with respect to the thickness coordinate, now the antisymmetric solution is sought. Because of the great formal similarity between the theory in Chapter I and the theory to be elaborated here, the developments in this chapter may be kept somewhat shorter.

The system of coordinates is set up in the same way as in Chapter I, and the problem to be solved now is formulated by equations (1.1) with boundary conditions (1.21) and (1.23), while boundary condition (1.22) is replaced by

$$\sigma_x = T\frac{z}{h}; \quad \tau_{xy} = \sigma_y = 0, \text{ for } |x| \to \infty. \tag{3.1}$$

The loading moment M_0 per unit of length at infinity is

$$M_0 = \int_{-h}^{+h} \sigma_x z\, dz = -\frac{2}{3} T h^2. \qquad (3.2)$$

3.2 Elementary Stress Distributions

We start with the solution for the plate without a hole, in wich $\sigma_z = T\frac{z}{h}$ and all the residual stresses are zero. In cylindrical coordinates we have

$$\left.\begin{aligned} \frac{\sigma_r}{T} &= \frac{1}{2}\frac{z}{h}(1+\cos 2\varphi), \\ \frac{\sigma_\varphi}{T} &= \frac{1}{2}\frac{z}{h}(1-\cos 2\varphi), \\ \frac{\tau_{r\varphi}}{T} &= -\frac{1}{2}\frac{z}{h}(\sin 2\varphi), \\ \sigma_z &= \tau_{rz} = \tau_{\varphi z} = 0. \end{aligned}\right] \qquad (3.3)$$

The remaining solutions are derived from the potentials. The Laplace equation in cylindrical coordinates (1.25) is /66

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} = 0. \qquad (3.4)$$

First examined is the stress distribution corresponding to $B_2 = c_1 z \log r$, in which the set of constants $\alpha = 1 - 2\nu$ and $\gamma = -2(1 - \nu)$ defined by expression (1.12) is now used. From expressions (1.15) and (1.16) it then follows that

$$\left.\begin{aligned} \frac{\sigma_r}{2G} &= c_1 . -2(1-\nu)\frac{z}{r^2}, \\ \frac{\sigma_\varphi}{2G} &= c_1 . 2(1-\nu)\frac{z}{r^2}, \\ \sigma_z &= \tau_{z\varphi} = \tau_{rz} = \tau_{r\varphi} = 0. \end{aligned}\right] \qquad (3.5)$$

Then the stress distribution is studied which belongs to $A = c_2 \frac{z}{r^2} \sin 2\phi$; from expression (1.18) it then follows that /67

$$\left.\begin{aligned} \frac{\sigma_r}{2G} &= c_2 . -6\frac{z}{r^4}\cos 2\varphi, \\ \frac{\sigma_\varphi}{2G} &= c_2 . 6\frac{z}{r^4}\cos 2\varphi, \\ \sigma_z &= 0, \\ \frac{\tau_{\varphi z}}{2G} &= c_2 . \frac{1}{r^3}\sin 2\varphi, \end{aligned}\right] \qquad (3.6)$$

$$\frac{\tau_{rz}}{2G} = c_2 \cdot \frac{1}{r^3} \cos 2\varphi,$$

$$\frac{\tau_{r\varphi}}{2G} = c_2 \cdot -6 \frac{z}{r^4} \sin 2\varphi \qquad (3.6)$$

With potential $B_1 = c_3 \frac{z}{r^2} \cos 2\phi$, a third stress system /67

$$\frac{\sigma_r}{2G} = c_3 \cdot 6 \frac{z}{r^4} \cos 2\varphi,$$

$$\frac{\sigma_\varphi}{2G} = c_3 \cdot -6 \frac{z}{r^4} \cos 2\varphi,$$

$$\sigma_z = 0. \qquad (3.7)$$

$$\frac{\tau_{\varphi z}}{2G} = c_3 \cdot -\frac{2}{r^3} \sin 2\varphi,$$

$$\frac{\tau_{rz}}{2G} = c_3 \cdot -\frac{2}{r^3} \cos 2\varphi,$$

$$\frac{\tau_{r\varphi}}{2G} = c_3 \cdot 6 \frac{z}{r^4} \sin 2\varphi.$$

is found by means of equations (1.19). It is noted that a suitable combination of expressions (3.6) and (3.7) provides a plane state of stress.

A solution of expression (3.4) which will also be used is /68

$$B_2 = \left(\frac{3z}{2} + \frac{z^3}{r^2}\right) \cos 2\varphi, \text{ met } \alpha = 1 - 2\nu \text{ en } \gamma = -2(1-\nu).$$

From this the stress distribution

$$\frac{\sigma_r}{2G} = c_4 \left[12(2-\nu)\frac{z^3}{r^4} - \frac{12\nu z}{r^2}\right] \cos 2\varphi,$$

$$\frac{\sigma_\varphi}{2G} = c_4 \left[-12(2-\nu)\frac{z^3}{r^4} - \frac{12z}{r^2}\right] \cos 2\varphi,$$

$$\sigma_z = 0.$$

$$\frac{\tau_{\varphi z}}{2G} = c_4 \cdot -12 \frac{z^2}{r^3} \sin 2\varphi, \qquad (3.8)$$

$$\frac{\tau_{rz}}{2G} = c_4 \cdot -12 \frac{z^2}{r^3} \cos 2\varphi,$$

$$\frac{\tau_{r\varphi}}{2G} = c_4 \left[12(2-\nu)\frac{z^3}{r^4} + 6(1-\nu)\frac{z}{r^2}\right] \sin 2\varphi.$$

follows. The following choice is now made for the constants c_1, c_2, c_3, and c_4:

$$\left.\begin{aligned}
c_1 &= \frac{T}{2G}\,\frac{1}{4(1-\nu)}\,\frac{a^2}{h},\\
c_2 &= \frac{T}{2G}\left(\frac{a^4}{h}\,2C - a^2hD\right),\\
c_3 &= \frac{T}{2G}\,\frac{a^4}{h}\,C,\\
c_4 &= -\frac{T}{2G}\,\frac{1}{12}\,\frac{a^2}{h}\,D,
\end{aligned}\right] \tag{3.9}$$

in which C and D are to be determined still more exactly.

Superposition of system (3.3), (3.5), (3.6), (3.7), and (3.8) now /69 gives the elementary stress distribution

$$\left.\begin{aligned}
\frac{\sigma_r}{T} &= \frac{1}{2}\,\frac{z}{h}\left(1-\frac{a^2}{r^2}\right) + \cos 2\varphi\,\frac{z}{h}\left\{\frac{1}{2} - \right.\\
&\quad \left. -6C\frac{a^4}{r^4} + D\left(6\frac{a^2h^2}{r^4} + \frac{\nu a^2}{r^2} - (2-\nu)\frac{a^2z^2}{r^4}\right)\right\},\\
\frac{\sigma_\varphi}{T} &= \frac{1}{2}\,\frac{z}{h}\left(1+\frac{a^2}{r^2}\right) + \cos 2\varphi\,\frac{z}{h}\left\{-\frac{1}{2} + \right.\\
&\quad \left. +6C\frac{a^4}{r^4} + D\left(-6\frac{a^2h^2}{r^4} + \frac{a^2}{r^2} + (2-\nu)\frac{a^2z^2}{r^4}\right)\right\},\\
\sigma_z &= 0,\\
\frac{\tau_{\varphi z}}{T} &= \sin 2\varphi\,\frac{a^2}{h}\,D\left(\frac{z^2}{r^3} - \frac{h^2}{r^3}\right),\\
\frac{\tau_{rz}}{T} &= \cos 2\varphi\,\frac{a^2}{h}\,D\left(\frac{z^2}{r^3} - \frac{h^2}{r^3}\right),\\
\frac{\tau_{r\varphi}}{T} &= \sin 2\varphi\,\frac{z}{h}\left\{-\frac{1}{2} - 6C\frac{a^4}{r^4} + \right.\\
&\quad \left. + D\left(6\frac{a^2h^2}{r^4} - \frac{1-\nu}{2}\,\frac{a^2}{r^2} - (2-\nu)\frac{a^2z^2}{r^4}\right)\right\}.
\end{aligned}\right] \tag{3.10}$$

The stress distribution (3.10) satisfies the requirement that the plate's boundary planes ($z = \pm h$) be unstressed. For a *thin plate*, i.e., $\frac{h}{a} \to 0$, the boundary loads along the edge of the hole are small if

$$\left.\begin{aligned}
C &= \frac{1-3\nu}{12(1+\nu)},\\
D &= \frac{-2}{1+\nu}
\end{aligned}\right] \tag{3.11}$$

and although the boundary loads σ_r and $\tau_{r\phi}$ are of the order of magnitude /70
$\frac{h^2}{a^2}$, τ_{rz} along the boundary is of order $\frac{h}{a}$. Nevertheless, this elementary stress distribution *cannot* be considered the limiting case which is approached by the stress distribution if $\frac{h}{a} \to 0$. In order to show this it is noted that the (small) boundary loads τ_{rz} together form an equilibrium system along the whole boundary when r = a. On the other hand, they also cause the plate-half $0 < \phi < \pi$ to exert a torque on a plate-half $0 > \phi > -\pi$ having the magnitude

$$\int_0^\pi d\varphi \int_{-h}^{h} dz \,.\, a^2 \sin\varphi \,.\, \tau_{rz} = -\frac{16}{9(1+\nu)} \,.\, T \,.\, ah^2. \tag{3.12}$$

This torque is transmitted through the vicinity of the hole. The corresponding bending stress has an order of magnitude defined by equation (3.12) divided by ah^2, the resisting moment of a flat strip $^3/_2$ a wide, i.e., of the order of magnitude T. The disturbance of the elementary stress distribution (3.10), which is coupled with relief of boundary load τ_{rz}, consequently results in extra stresses (σ_ϕ, $\tau_{r\phi}$, σ_r) which are of *the same order of magnitude* as bending stress T at infinity.

Just as in the tension loaded plate, non-elementary stress distributions, which make the edge free of stress, are added to the stress distribution according to (3.10). In contrast to the tension-loaded plate these non-elementary stress distributions are here already essential for a very thin plate.

3.3 The Eigen Functions /71

Just as in Chapter I, it is also possible here to find stress distributions by means of potentials A, B_1, B_2, which leaves the boundary planes $z = +h$ free of stress. Moreover, we would like to find solutions of expression (3.4) of the type

$$f(r, \varphi, z) = K_2(\lambda r) \left\{ \begin{matrix} \cos\lambda z \\ \sin\lambda z \end{matrix} \right\} \left\{ \begin{matrix} \cos 2\varphi \\ \sin 2\varphi \end{matrix} \right\}. \tag{3.13}$$

From system (3.10) it follows that the choice of $\cos\lambda z$ or $\sin\lambda z$ for the potential functions must be done in such a way that σ_r, σ_ϕ, σ_z, and $\tau_{r\phi}$ become antisymmetrical functions of z and τ_{rz} and $\tau_{\phi z}$ become symmetrical functions.

For potential function A, we now choose

$$A = \frac{Ta^2}{2G} K_2(\lambda r) \sin\lambda z \sin 2\varphi, \tag{3.14}$$

It follows in accord with expression (1.18), when $|z| = h$, that

$$\left.\begin{aligned} \sigma_z &= 0, \\ \frac{\tau_{\varphi z}}{T} &= -\frac{a^2\lambda^2}{2} K'_2(\lambda r) \cos\lambda h \sin 2\varphi, \end{aligned}\right] \tag{3.15}$$

$$\frac{\tau_{rz}}{T} = \frac{a^2\lambda}{r} K_2(\lambda r) \cos \lambda h \cos 2\varphi. \tag{3.15}$$

The boundary surface is unstressed if parameter λ is so chosen that $\cos \lambda h = 0$, thus

$$\lambda_k = \frac{k'\pi}{h}, \tag{3.16}$$

$$\text{with } k' = k + \frac{1}{2}. \qquad (k = 0, 1, 2 \ldots) \tag{3.17}$$

By linear combination, we obtain /72

$$A = \frac{Ta^2}{2G} \sum_{k=0}^{\infty} a_k \frac{K_2\left(\frac{k'\pi r}{h}\right)}{K_2\left(\frac{k'\pi a}{h}\right)} \sin \frac{k'\pi z}{h} \sin 2\varphi. \tag{3.18}$$

Potentials B_1 and B_2 are similarly treated. If it is assumed that

$$\left.\begin{aligned} B_1 &= B_1^* \frac{Ta^2}{2G} K_2(\lambda r) \sin \lambda z \cos 2\varphi, \\ B_2 &= B_2^* \frac{Ta^2}{2G} K_2(\lambda r) \sin \lambda z \cos 2\varphi, \end{aligned}\right] \tag{3.19}$$

from expressions (1.19) and (1.20) when $|z| = h$, we derive

$$\left.\begin{aligned} \frac{\sigma_z}{T} &= \pm a^2 \cos 2\varphi K_2(\lambda r) \; [-B_1^*\lambda^2 \sin \lambda h - \\ &\qquad - B_2^*\lambda^3 h \cos \lambda h], \\ \frac{\tau_{\varphi z}}{T} &= -\frac{2a^2}{r} \cos 2\varphi K_2(\lambda r) \; [B_1^*\lambda \cos \lambda h + \\ &\qquad + B_2^*\lambda \cos \lambda h - B_2^*\lambda^2 h \sin \lambda h], \\ \frac{\tau_{rz}}{T} &= a^2 \cos 2\varphi K_2'(\lambda r) \; [B_1^*\lambda^2 \cos \lambda h + B_2^*\lambda^2 \cos \lambda h - \\ &\qquad - B_2^*\lambda^3 h \sin \lambda h]. \end{aligned}\right] \tag{3.20}$$

For all values of (r, ϕ) these stresses are zero if

$$\left.\begin{aligned} &B_1^* \sin \lambda h + B_2^* \lambda h \cos \lambda h = 0, \\ &B_1^* \cos \lambda h + B_2^* (\cos \lambda h - \lambda h \sin \lambda h) = 0, \end{aligned}\right] \tag{3.21}$$

The preceding system has solutions differing from zero only if the determinant is zero, i.e., when λ satisfies the equation /73

$$\sin 2\lambda h - 2\lambda h = 0, \tag{3.22}$$

while the relationship between B_1^* and B_2^* is then defined by

$$B_1^* = -B_2^* \lambda h \cot \lambda h. \tag{3.23}$$

Eigen value equation (3.22) again has a four-fold infinite series of solutions. If λ_ℓ is an eigen value, then $-\lambda_\ell$, $\bar{\lambda}_\ell$, and $-\bar{\lambda}_\ell$ are also eigen values. Here, too, the examination may be limited to the eigen values for which

$$\left.\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\right\} \lambda_l > 0. \tag{3.24}$$

By linear combination, we obtain

$$\left.\begin{aligned} B_1 &= \frac{Ta^2}{2G} \mathrm{Re} \sum_{l=1}^{\infty} b_l \left(-\lambda_l h \cot \lambda_l h\right) \frac{K_2(\lambda_l r)}{K_2(\lambda_l a)} \sin \lambda_l z \cos 2\varphi. \\ B_2 &= \frac{Ta^2}{2G} \mathrm{Re} \sum_{l=1}^{\infty} b_l \frac{K_2(\lambda_l r)}{K_2(\lambda_l a)} \sin \lambda_l z \cos 2\varphi. \end{aligned}\right] \tag{3.25}$$

After introduction of the dimensionless coordinates, in accord with expression (1.52), the system with the eigen value equation (the asterisks are again left off) becomes

$$\left.\begin{aligned} A &= \frac{Ta^2}{2G} \sum_{k=0}^{\infty} a_k \frac{K_2(k'\pi\beta r)}{K_2(k'\pi\beta)} \sin k'\pi z \sin 2\varphi, \\ B_1 &= \frac{Ta^2}{2G} \mathrm{Re} \sum_{l=1}^{\infty} b_l \left(-\lambda_l \cot \lambda_l\right) \\ &\qquad \frac{K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)} \sin \lambda_l z \cos 2\varphi, \\ B_2 &= \frac{Ta^2}{2G} \mathrm{Re} \sum_{l=1}^{\infty} b_l \frac{K_2(\lambda_l \beta r)}{K_2(\lambda_l \beta)} \sin \lambda_l z \cos 2\varphi, \\ \sin 2\lambda_l &= 2\lambda_l. \end{aligned}\right] \tag{3.26}$$

3.4 <u>Stresses Along the Boundary r = 1</u>

In order to be able to fulfill the boundary conditions for r = 1, the stress system (3.10) and the systems derived from expression (3.26) are superposed. In dimensionless coordinates, system (3.10) becomes /74

$$\left.\begin{aligned} \frac{\sigma_r}{T} &= \frac{1}{2} z\left(1-\frac{1}{r^2}\right) + \cos 2\varphi \cdot z \left\{\frac{1}{2} - 6C\frac{1}{r^4} + \right. \\ &\quad \left. + D\left(\frac{\nu}{r^2} + \frac{6}{\beta^2 r^4} - (2-\nu)\frac{z^2}{\beta^2 r^4}\right)\right\}, \\ \frac{\sigma_\varphi}{T} &= \frac{1}{2} z\left(1+\frac{1}{r^2}\right) + \cos 2\varphi \cdot z \left\{-\frac{1}{2} + \right. \\ &\quad \left. + 6C\frac{1}{r^4} + D\left(\frac{1}{r^2} - \frac{6}{\beta^2 r^4} + (2-\nu)\frac{z^2}{\beta^2 r^4}\right)\right\}, \\ \sigma_z &= 0. \end{aligned}\right] \tag{3.27}$$

$$\left.\begin{aligned}
\frac{\tau_{\varphi z}}{T} &= \sin 2\varphi \frac{D}{\beta r^3}(z^2-1),\\
\frac{\tau_{rz}}{T} &= \cos 2\varphi \frac{D}{\beta r^3}(z^2-1),\\
\frac{\tau_{r\varphi}}{T} &= \sin 2\varphi \,.\, z \left\{-\frac{1}{2} - 6C\frac{1}{r^4} + \right.\\
&\left. + D\left(-\frac{1-\nu}{2}\frac{1}{r^2} + \frac{6}{\beta^2 r^4} - (2-\nu)\frac{z^2}{\beta^2 r^4}\right)\right\}.
\end{aligned}\right] \qquad (3.27)$$

When r = 1, this gives the boundary load

$$\left.\begin{aligned}
\frac{\sigma_r^{(2)}}{T} &= z\left\{\frac{1}{2} - 6C + D\left(\nu + \frac{6}{\beta^2} - (2-\nu)\frac{z^2}{\beta^2}\right)\right\},\\
\frac{\tau_{rz}^{(2)}}{T} &= \frac{D}{\beta}(z^2-1),\\
\frac{\tau_{r\varphi}^{(2)}}{T} &= z\left\{-\frac{1}{2} - 6C + D\left(-\frac{1-\nu}{2} + \frac{6}{\beta^2} - (2-\nu)\frac{z^2}{\beta^2}\right)\right\},
\end{aligned}\right] \qquad (3.28)$$

where the superscript (2) has the same meaning as in expression (1.55). By /75 means of the expansions which are valid in the $-1 < z < 1$ range

$$z = 2\sum_{k=0}^{\infty}\frac{(-1)^k \sin k'\pi z}{k'^2\pi^2}, \quad (-1 \leqq z \leqq 1), \qquad (3.29)$$

$$z^2 - 1 = -4\sum_{k=0}^{\infty}\frac{(-1)^k \cos k'\pi z}{k'^3\pi^3}, \quad (-1 < z < 1), \qquad (3.30)$$

$$\left.\begin{aligned}
z^3 &= 6\sum_{k=0}^{\infty}\frac{(-1)^k \sin k'\pi z}{k'^2\pi^2} -\\
&- 12\sum_{k=0}^{\infty}\frac{(-1)^k \sin k'\pi z}{k'^4\pi^4}, \quad (-1 \leqq z \leqq 1),
\end{aligned}\right] \qquad (3.31)$$

we can write the following:

$$\left.\begin{aligned}
\frac{\sigma_r^{(2)}}{T} &= \sum_{k=0}^{\infty}(-1)^k \sin k'\pi z \left\{\frac{1}{k'^2\pi^2} - \frac{12C}{k'^2\pi^2} + \right.\\
&\left. + D\left(\frac{2\nu + \frac{6\nu}{\beta^2}}{k'^2\pi^2} + \frac{12(2-\nu)}{\beta^2 k'^4\pi^4}\right)\right\},\\
\frac{\tau_{rz}^{(2)}}{T} &= -\frac{4}{\beta}D\sum_{k=0}^{\infty}(-1)^k\frac{\cos k'\pi z}{k'^3\pi^3},
\end{aligned}\right] \qquad (3.32)$$

$$\frac{\tau_{r\varphi}^{(2)}}{T} = \sum_{k=0}^{\infty} (-1)^k \sin k'\pi z \Big\{ -\frac{1}{k'^2\pi^2} - \frac{12\,C}{k'^2\pi^2} + + D\left(\frac{-(1-\nu) + \frac{6\nu}{\beta^2}}{k'^2\pi^2} + \frac{12(2-\nu)}{\beta^2 k'^4 \pi^4}\right)\Big\}. \tag{3.32}$$

By means of potentials (3.26), the other stress systems may also be derived. After some computations, it is found that when r = 1 /76

$$\begin{aligned}
\frac{\sigma_r^{(2)}}{T} &= \sum_{k=0}^{\infty} a_k p_k^{(1)} \sin k'\pi z + \mathrm{Re} \sum_{l=1}^{\infty} b_l q_l^{(1)} \sin \lambda_l z + \\
&\qquad + \mathrm{Re} \sum_{l=1}^{\infty} b_l r_l^{(1)} \lambda_l z \cos \lambda_l z, \\
\frac{\tau_{rz}^{(2)}}{T} &= \beta \Big[\sum_{k=0}^{\infty} a_k p_k^{(2)} \cos k'\pi z + \\
&+ \mathrm{Re} \sum_{l=1}^{\infty} b_l q_l^{(2)} \cos \lambda_l z + \mathrm{Re} \sum_{l=1}^{\infty} b_l r_l^{(2)} \lambda_l z \sin \lambda_l z \Big], \\
\frac{\tau_{r\varphi}^{(2)}}{T} &= \sum_{k=0}^{\infty} a_k p_k^{(3)} \sin k'\pi z + \mathrm{Re} \sum_{l=1}^{\infty} b_l q_l^{(3)} \sin \lambda_l z + \\
&\qquad + \mathrm{Re} \sum_{l=1}^{\infty} b_l r_l^{(3)} \lambda_l z \cos \lambda_l z,
\end{aligned} \tag{3.33}$$

in which the following abbreviations are used

$$\begin{aligned}
p_k^{(1)} &= 2\,K(k'\pi\beta) - 2, \\
p_k^{(2)} &= k'\pi, \\
p_k^{(3)} &= K(k'\pi\beta) - \frac{(k'\pi\beta)^2}{2} - 4;
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
q_l^{(1)} &= (2(1-\nu) - \lambda_l \cot \lambda_l)\ (-K(\lambda_l\beta) + 4) + \\
&\qquad + \lambda^2{}_l \beta^2 (2 - \lambda_l \cot \lambda_l), \\
q_l^{(2)} &= \lambda_l (1 - \lambda_l \cot \lambda_l)\ K(\lambda_l\beta), \\
q_l^{(3)} &= -2(2(1-\nu) - \lambda_l \cot \lambda_l)\ (K(\lambda_l\beta - 1);
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
r_l^{(1)} &= -\,(K(\lambda_l\beta) - \lambda^2{}_l\beta^2 - 4), \\
r_l^{(2)} &= -\lambda_l K(\lambda_l\beta), \\
r_l^{(3)} &= -2(K(\lambda_l\beta) - 1).
\end{aligned} \tag{3.36}$$

Further computations take a course similar to that treated in Chapter I.

Stresses (3.33) are expanded in series with respect to sin $k'\pi z$ or cos $k'\pi z$ /77
and, together with stresses (3.30) which are assumed to be zero, are set equal to zero by setting all Fourier coefficients equal to zero. This results in the following system of equations

$$\left.\begin{aligned}
&(-1)^k a_k p_k^{(1)} + \mathrm{Re}\sum_{l=1}^{\infty} b_l \psi_{lk}^{(1)} + C\, m_k + D\, n_k^{(1)} + Q_k^{(1)} = 0,\\
&(-1)^k a_k p_k^{(2)} + \mathrm{Re}\sum_{l=1}^{\infty} b_l \psi_{lk}^{(2)} + \qquad + D\, n_k^{(2)} = 0,\\
&(-1)^k a_k p_k^{(3)} + \mathrm{Re}\sum_{l=1}^{\infty} b_l \psi_{lk}^{(3)} + C\, m_k + D\, n_k^{(3)} + Q_k^{(3)} = 0,\\
&\qquad (k = 0,\ 1,\ 2,\ \ldots)
\end{aligned}\right] \qquad (3.37)$$

in which the newly introduced quantities are defined by

$$\left.\begin{aligned}
\psi_{lk}^{(1)} &= 4\beta^2\lambda^3_l \cos\lambda_l \frac{k'^2\pi^2}{(\lambda^2_l - k'^2\pi^2)^2} -\\
&- 4\lambda_l\cos\lambda_l \frac{1}{\lambda^2_l - k'^2\pi^2}\cdot\left[\nu + \frac{k'^2\pi^2}{\lambda^2_l - k'^2\pi^2}\right]\cdot[K(\lambda_l\beta) - 4],\\
\psi_{lk}^{(2)} &= 4\lambda^3_l\cos\lambda_l \frac{k'\pi}{(\lambda^2_l - k'^2\pi^2)^2} K(\lambda_l\beta),\\
\psi_{lk}^{(3)} &= -8\lambda_l\cos\lambda_l \frac{1}{\lambda^2_l - k'^2\pi^2}\left[\nu + \right.\\
&\qquad \left. + \frac{k'^2\pi^2}{\lambda^2_l - k'^2\pi^2}\right][K(\lambda_l\beta) - 1];\\
&\qquad (k = 0,\ 1,\ 2\ldots)\\
&\qquad (l = 1,\ 2\ldots)
\end{aligned}\right] \qquad (3.38)$$

$$\left.\begin{aligned}
m_k &= \frac{-12}{k'^2\pi^2};\\
n_k^{(1)} &= \left[\frac{2\nu + \dfrac{6\nu}{\beta^2}}{k'^2\pi^2} + \frac{12(2-\nu)}{\beta^2 k'^4\pi^4}\right],\\
n_k^{(2)} &= -\frac{4}{\beta^2}\,\frac{1}{k'^3\pi^3},\\
n_k^{(3)} &= \left[\frac{-(1-\nu) + \dfrac{6\nu}{\beta^2}}{k'^2\pi^2} + \frac{12(2-\nu)}{\beta^2 k'^4\pi^4}\right];\\
Q_k^{(1)} &= \frac{1}{k'^2\pi^2},
\end{aligned}\right] \qquad (3.39)$$

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$$Q_k^{(3)} = \frac{-1}{k'^2\pi^2}. \quad (3.39)$$

From the second equation we have

$$a_k = -\frac{(-1)^k}{p_k^{(2)}} \left\{ \mathrm{Re} \sum_{l=1}^{\infty} b_l \psi_{lk}^{(2)} + D n_k^{(2)} \right\} \quad (3.40)$$

wherewith the remaining equations are converted into

$$\left.\begin{aligned} &\mathrm{Re} \sum_{l=1}^{\infty} b_l \chi_{lk}^{(1)} + C m_k + D \left(n_k^{(1)} - \frac{p_k^{(1)}}{p_k^{(2)}} n_k^{(2)} \right) + Q_k^{(1)} = 0, \\ &\mathrm{Re} \sum_{l=1}^{\infty} b_l \chi_{lk}^{(2)} + C m_k + D \left(n_k^{(3)} - \frac{p_k^{(3)}}{p_k^{(2)}} n_k^{(2)} \right) + Q_k^{(3)} = 0, \\ &(k = 0, 1, 2, \ldots) \end{aligned}\right] \quad (3.41)$$

where /79

$$\left.\begin{aligned} \chi_{lk}^{(1)} &= \psi_{lk}^{(1)} - \frac{p_k^{(1)}}{p_k^{(2)}} \psi_{lk}^{(2)}, \\ \chi_{lk}^{(2)} &= \psi_{lk}^{(3)} - \frac{p_k^{(3)}}{p_k^{(2)}} \psi_{lk}^{(2)}. \end{aligned}\right] \quad (3.42)$$

A simplification in the numerical calculation is obtained by introducing

$$\chi_{lk}^{*(n)} = \frac{\chi_{lk}^{(n)}}{4\lambda^3{}_l \cos \lambda_l}, \quad (3.43)$$

$$b_l = \frac{c_l}{4\lambda^3{}_l \cos \lambda_l}, \quad (3.44)$$

by which the system assumes the following form

$$\left.\begin{aligned} &\mathrm{Re} \sum_{l=1}^{\infty} c_l \chi_{lk}^{*(1)} + C m_k + D \left(n_k^{(1)} - \frac{p_k^{(1)}}{p_k^{(2)}} n_k^{(2)} \right) + Q_k^{(1)} = 0, \\ &\mathrm{Re} \sum_{l=1}^{\infty} c_l \chi_{lk}^{*(2)} + C m_k + D \left(n_k^{(3)} - \frac{p_k^{(3)}}{p_k^{(2)}} n_k^{(2)} \right) + Q_k^{(3)} = 0. \\ &(k = 0, 1, 2, \ldots) \end{aligned}\right] \quad (3.45)$$

Just as in Chapter I, the problem is now reduced to a twofold, infinite system of equations with twofold infinite unknowns C, D, c_1, c_2, ... For practical computations, system (3.45) is truncated to a finite system. The general remarks of Section 1.7, in regard to the approximate solution of system (1.73), (1.74), are of similar pertinence here. It must, however, be noted that system (1.73), (1.74) is not analogous in structure to system (3.37). While it is clear that system (1.74) is to be interpreted as equations to determine coefficients a_k, b_ℓ expressed in D -- after which expression (1.73) is used to calculate C and D -- a similar interpretation of system (3.37) is not obvious. Undoubtedly, this difference is related to the chosen type of Fourier expansion over sin $k'\pi z$ and cos $k'\pi z$, where now, in contrast to

Chapter I, the value $k = 0$ occupies no particular place.

3.5 The Thin Plate

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If, in the extreme case of the infinitely thin plate ($\beta \to \infty$), we set c_ℓ equal to zero, system (3.45) degenerates into two equations for C and D with the solutions

$$C = \frac{1-\nu}{4(3+\nu)} \; ; \quad D = -\frac{2}{3+\nu}. \tag{3.46}$$

The pertinent stress distribution agrees with the solution determined by Goodier for the classical plate equation involving a plate with a circular hole (Ref. 8).

It is clear that for a plate of slight, but finite, thickness the solution is to be taken in the form of an asymptotic series. In this asymptotic expansion, it is advisable to proceed from system (3.37) whose coefficients $p_k^{(n)}$, $\psi_{\ell k}^{(n)}$, $n = 1, 2, \ldots$ are developed as follows:

$$\left.\begin{aligned} p_k^{(1)} &= -2k'\pi\beta - 3 - \frac{15}{4}\,\frac{1}{k'\pi\beta} + \ldots, \\ p_k^{(2)} &= k'\pi, \\ p_k^{(3)} &= -\frac{(k'\pi\beta)^2}{2} - k'\pi\beta - \frac{9}{2} + \ldots, \end{aligned}\right] \tag{3.47}$$

$$\left.\begin{aligned} \psi_{lk}^{(1)} &= \beta^2 . 4\lambda_l^3 \cos\lambda_l . \frac{k'^2\pi^2}{(\lambda_l^2 - k'^2\pi^2)^2} + \ldots, \\ \psi_{lk}^{(2)} &= -\beta . 4\lambda_l^4 \cos\lambda_l \frac{k'\pi}{(\lambda_l^2 - k'^2\pi^2)^2} + \ldots, \\ \psi_{lk}^{(3)} &= \beta . 8\lambda_l^2 \cos\lambda_l \frac{1}{(\lambda_l^2 - k'^2\pi^2)}\left(\nu + \frac{k'^2\pi^2}{\lambda_l^2 - k'^2\pi^2}\right) + \ldots \end{aligned}\right] \tag{3.48}$$

It is now immediately evident that the first terms of the desired asymptotic expansion are

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$$\left.\begin{aligned} a_k &= \frac{a_k^{(2)}}{\beta^2} + \frac{a_k^{(3)}}{\beta^3} + \ldots, \\ b_l &= \frac{b_l^{(3)}}{\beta^3} + \frac{b_l^{(4)}}{\beta^4} + \ldots, \\ C &= C^{(0)} + \frac{C^{(1)}}{\beta} + \ldots, \\ D &= D^{(0)} + \frac{D^{(1)}}{\beta} + \ldots. \end{aligned}\right] \tag{3.49}$$

Substitution of this into (3.37) and ordering by ascending powers of $\frac{1}{\beta}$ leads to the equations where k = 0, 1, 2, ...

$$12\,C^{(0)} - 2\nu\,D^{(0)} = 1, \tag{3.50}$$

$$\left.\begin{aligned} & -2\,k'\pi\,(-1)^k a_k^{(2)} + \mathrm{Re}\sum_{l=1}^{\infty} 4\,b_l^{(3)}\,\lambda_l^3 \cos\lambda_l \frac{k'^2\pi^2}{(\lambda_l^2 - k'^2\pi^2)^2} - \\ & \qquad - \frac{12\,C^{(1)}}{k'^2\pi^2} + \frac{2\nu\,D^{(1)}}{k'^2\pi^2} = 0, \end{aligned}\right] \tag{3.51}$$

$$\left.\begin{aligned} & (-1)^k a_k^{(2)} + \mathrm{Re}\sum_{l=1}^{\infty} -4\,\lambda_l^4 \cos\lambda_l\, b_l^{(3)} \frac{1}{(\lambda_l^2 - k'^2\pi^2)^2} - \\ & \qquad - \frac{4}{k'^4\pi^4}\,D^{(0)} = 0, \end{aligned}\right] \tag{3.52}$$

$$\left.\begin{aligned} & (-1)^k a_k^{(3)} + \mathrm{Re}\sum_{l=1}^{\infty} -4\,\lambda_l^4 \cos\lambda_l\, b_l^{(4)} \frac{1}{(\lambda_l^2 - k'^2\pi^2)^2} + \\ & + \mathrm{Re}\sum_{l=1}^{\infty} -2\,\lambda_l^3 \cos\lambda_l\, b_l^{(3)} \frac{1}{(\lambda_l^2 - k'^2\pi^2)^2} - \frac{4}{k'^4\pi^4}\,D^{(1)} = 0, \end{aligned}\right] \tag{3.53}$$

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$$(-1)^k a_k^{(2)} + \frac{24}{k'^4\pi^4}\,C^{(0)} + \frac{2(1-\nu)}{k'^4\pi^4}D^{(0)} = -\frac{2}{k'^4\pi^4}, \tag{3.54}$$

$$\left.\begin{aligned} & (-1)^k a_k^{(3)} + \frac{2}{k'\pi}(-1)^k a_k^{(2)} + \\ & \qquad + \frac{24}{k'^4\pi^4}\,C^{(1)} + \frac{2(1-\nu)}{k'^4\pi^4}D^{(1)} = 0. \end{aligned}\right] \tag{3.55}$$

By means of (3.54) and (3.55) $a_k^{(2)}$ and $a_k^{(3)}$ are expressed in $C^{(0)}$, $D^{(0)}$, $C^{(1)}$, and $D^{(1)}$.

$$a_k^{(2)} = \frac{(-1)^{k+1}}{k'^4\pi^4}\,[2 + 24\,C^{(0)} + 2(1-\nu)D^{(0)}], \tag{3.56}$$

$$\left.\begin{aligned} a_k^{(3)} = & \frac{(-1)^{k+1}}{k'^4\pi^4}[24\,C^{(1)} + 2\,(1-\nu)D^{(1)}] + \\ & + \frac{4\,.\,(-1)^k}{k'^5\pi^5}[1 + 12\,C^{(0)} + (1-\nu)D^{(0)}]. \end{aligned}\right] \tag{3.57}$$

Expressions (3.56) and (3.57) are now used to derive three equations, in which only $C^{(0)}$, $D^{(0)}$, $C^{(1)}$ and $D^{(1)}$ occur as unknowns, from equations (3.51), (3.52) and (3.53) by eliminating b_ℓ. Together with expression (3.50), these equations determine the unknowns mentioned. Elimination of b_ℓ is achieved by means of the identity

$$\sum_{k=0}^{\infty} \frac{1}{(\lambda_l^2 - k'^2\pi^2)^2} = 0, \tag{3.58}$$

which is true for all *eigen values* λ_ℓ. Identity (3.58) is obtained from the series expansions in the interval $-1 \leq z \leq 1$ for all arbitrary values of λ

$$\sin \lambda z = \sum_{k=0}^{\infty} 2(-1)^{k+1} \frac{\lambda \cos \lambda}{\lambda^2 - k'^2\pi^2} \sin k'\pi z, \tag{3.59}$$

$$\lambda z \cos \lambda z = \sum_{k=0}^{\infty} 2(-1)^k \lambda \cos \lambda \left[\frac{\lambda^2 + k'^2\pi^2}{(\lambda^2 - k'^2\pi^2)^2} + \right. \left. + \frac{\lambda \operatorname{tg} \lambda}{\lambda^2 - k'^2\pi^2} \right] \sin k'\pi z, \tag{3.60}$$

by chosing z = 1 and making use of the eigen value equation /83

$$\sin 2\lambda_l - 2\lambda_l = 0.$$

With arbitrary values of λ, we also have from (3.59) the well-known expansion (Ref. 31)

$$\sum_{k=0}^{\infty} \frac{1}{\lambda^2 - k'^2\pi^2} = -{}^1/_2 \frac{\operatorname{tg} \lambda}{\lambda}, \tag{3.61}$$

while from expressions (3.58) and (3.61) for an *eigen value* λ_ℓ we have

$$\sum_{k=0}^{\infty} \frac{k'^2\pi^2}{(\lambda_l^2 - k'^2\pi^2)^2} = {}^1/_2 \frac{\operatorname{tg} \lambda_l}{\lambda_l}. \tag{3.62}$$

Elimination of b_ℓ from expression (3.51) is now achieved by both terms only by dividing by $k'^2\pi^2$ and then summing over k from zero to infinity. The result is the equation

$$\frac{124}{\pi^5} \cdot \zeta(5) \left[1 + 12\, C^{(0)} + (1-\nu) D^{(0)}\right] - {}^1/_3 \left(6\, C^{(1)} - \nu\, D^{(1)}\right) = 0, \tag{3.63}$$

where $\zeta(n)$ is Riemann's zeta function, and use is made of the relationship

$$\sum_{k=0}^{\infty} \frac{1}{k'^n} = (2^n - 1)\, \zeta(n). \tag{3.64}$$

Elimination of b_ℓ from expressions (3.52) and (3.53) is directly obtained by summation over k from zero to infinity

$$12\, C^{(0)} + (3-\nu) D^{(0)} = -1, \tag{3.65}$$

$$4C^{(1)} + \tfrac{1}{3}(3-\nu)D^{(1)} - \frac{124}{\pi^5}\zeta(5)\left[1 + 12C^{(0)} + (1-\nu)D^{(0)}\right] = 0. \qquad (3.66)$$ /84

The solution of the four equations (3.50), (3.65), (3.63) and (3.66) is

$$C^{(0)} = \frac{1-\nu}{4(3+\nu)}, \quad D^{(0)} = -\frac{2}{3+\nu}; \qquad (3.67)$$

$$\left.\begin{aligned} C^{(1)} &= \frac{744}{\pi^5}\zeta(5)\frac{1}{(3+\nu)^2}, \\ D^{(1)} &= -\frac{1488}{\pi^5}\zeta(5)\frac{1}{(3+\nu)^2}. \end{aligned}\right] \qquad (3.68)$$

By means of expressions (3.56) and (3.57), we may now calculate

$$a_k^{(2)} = (-1)^{k+1}\frac{8}{3+\nu}\,\frac{1}{k'^4\pi^4}, \qquad (3.69)$$

$$a_k^{(3)} = (-1)^k\frac{16}{3+\nu}\,\frac{1}{k'^4\pi^4}\left[\frac{1}{k'\pi} - \frac{186}{\pi^5}\zeta(5)\right]. \qquad (3.70)$$

The coefficients $b_\ell^{(3)}$ must finally be defined from the equations which follow from expressions (3.52) and (3.51) when k = 0, 1, 2, ...

$$\left.\begin{aligned} &\operatorname{Re}\sum_{l=1}^{\infty} b_l^{(3)}\frac{\lambda_l^4\cos\lambda_l}{(\lambda_l^2 - k'^2\pi^2)^2} = 0, \\ &\operatorname{Re}\sum_{l=1}^{\infty} b_l^{(3)}\frac{\lambda_l^3\cos\lambda_l}{(\lambda_l^2 - k'^2\pi^2)^2} = \\ &\qquad -\frac{4}{3+\nu}\,\frac{1}{k'^4\pi^4}\left[\frac{1}{k'\pi} - \frac{186}{\pi^5}\zeta(5)\right]. \end{aligned}\right] \qquad (3.71)$$

For future use, we should additionally note the relationship which is /85 obtained from the second equation (3.71) after multiplication by $k'^2\pi^2$ and summation over k from zero to infinity

$$\left.\begin{aligned} &\operatorname{Re}\sum_{l=1}^{\infty} b_l^{(3)}\,\lambda_l^2\sin\lambda_l = \\ &\qquad \frac{8}{\pi^3}\,\frac{1}{3+\nu}\left[\frac{93}{\pi^2}\zeta(5) - 7\,\zeta(3)\right]. \end{aligned}\right] \qquad (3.72)$$

3.6 The Moments, Transverse Forces, and Circumferential Stress

In the technical theory of the bending of thin plates, the bending moments M_r and M_ϕ, the torque $M_{r\phi}$ $(= M_{\phi r})$, and the transverse forces Q_r and Q_ϕ are introduced by integration of $z\cdot\sigma_r$, $z\cdot\sigma_\phi$, $z\tau_{r\phi} = z\tau_{\phi r}$, τ_{rz}, and $\tau_{\phi z}$ over the thickness. Finally, in order to obtain an equation with the classical theory, the formulas are derived for some of these quantities.

At the boundary r = 1, expressions (3.26) and (3.25) are used to find

$$\left(\frac{M_r^{(2)}}{M_0}\right)_{r=1} = \frac{h^2}{M_0} \int_{-1}^{1} \sigma_r^{(2)} \cdot z\,dz =$$
$$= 3\left[\sum_{k=0}^{\infty} a_k \cdot (-1)^k \cdot \frac{p_k^{(1)}}{k'^2\pi^2} + \right.$$
$$+ 2\nu \,\mathrm{Re} \sum_{l=1}^{\infty} b_l \cdot \frac{\sin\lambda_l}{\cos^2\lambda_l}\{K(\lambda_l\beta) - 4\} +$$
$$\left. + {}^1/_6 - 2C + D\left\{\frac{\nu}{3} + \frac{2}{\beta^2} - \frac{2-\nu}{5\beta^2}\right\}\right]; \qquad (3.73)$$

$$\left(\frac{Q_r^{(2)}}{Th}\right)_{r=1} = \int_{-1}^{1} \frac{\tau_{rz}^{(2)}}{T}\,dz =$$
$$= 2\beta\left[\sum_{k=0}^{\infty} a_k \cdot (-1)^k - {}^2/_3\,\frac{D}{\beta^2}\right]; \qquad (3.74)$$

$$\left(\frac{M_{r\varphi}^{(2)}}{M_0}\right)_{r=1} = \frac{h^2}{M_0} \int_{-1}^{1} \tau_{r\varphi}^{(2)} \cdot z \cdot dz =$$
$$= 3\left[\sum_{k=0}^{\infty} a_k \cdot (-1)^k \cdot \frac{p_k^{(3)}}{k'^2\pi^2} + \right.$$
$$+ 4\nu \,\mathrm{Re} \sum_{l=1}^{\infty} b_l \frac{\sin\lambda_l}{\cos^2\lambda_l}\{K(\lambda_l\beta) - 1\} -$$
$$\left. - {}^1/_6 - 2C + D\left\{-\frac{1-\nu}{6} + \frac{2}{\beta^2} - \frac{2-\nu}{5\beta^2}\right\}\right]. \qquad (3.75)$$

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Circumferential stress σ_ϕ is computed from (3.25) and (3.24) by means of expressions (1.18), (1.19), and (1.20). In dimensionless coordinates, σ_ϕ has the form

$$\frac{\sigma_\varphi}{T} = {}^1/_2\, z\left(1 + \frac{1}{r^2}\right) + \cos 2\varphi\left[z\left\{-{}^1/_2 + 6C\frac{1}{r^4} + \right.\right.$$
$$\left. + D\left(\frac{1}{r^2} - \frac{6}{\beta^2 r^4} - (2-\nu)\frac{z^2}{\beta^2 r^4}\right)\right\}$$
$$+ \frac{2}{r^2} \sum_{k=0}^{\infty} a_k \cdot \sin k'\pi z\left\{\frac{3K_2(k'\pi\beta r)}{K_2(k'\pi\beta)} - \right.$$
$$\left. - (k'\pi\beta r)\frac{K_1(k'\pi\beta r)}{K_2(k'\pi\beta)}\right\} + \mathrm{Re}\sum_{l=1}^{\infty} b_l \cdot \left\{\frac{1}{r^2}\,\frac{6K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)} - \right. \qquad (3.76)$$

$$-\frac{\lambda_l\beta}{r}\frac{K_1(\lambda_l\beta r)}{K_2(\lambda_l\beta)}\Big\{((\lambda_l\cot\lambda_l-2(1-\nu))\sin\lambda_l z-$$
$$-\lambda_l z\cos\lambda_l z\}+\mathrm{Re}\,2\nu\sum_{l=1}^{\infty}b_l\,.\,\lambda^2_l\beta^2\,.$$
$$\left.\cdot\,\frac{K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)}\sin\lambda_l z\right],\qquad(3.76)$$

which -- when r = 1 -- is simplified to /87

$$\left(\frac{\sigma_\varphi}{T}\right)_{r=1}=z+\cos 2\varphi\,[z\,.\,D(1+\nu)-$$
$$-\beta^2\,\mathrm{Re}\sum_{l=1}^{\infty}b_l\,.\,\lambda^3_l\cot\lambda_l\,.\,\sin\lambda_l z+2\,(1+\nu)\,\beta^2.\qquad(3.77)$$
$$\mathrm{Re}\sum_{l=1}^{\infty}b_l\lambda^2_l\sin\lambda_l z+\beta^2\,\mathrm{Re}\sum_{l=1}^{\infty}b_l\,.\,\lambda^3_l z\,.\,\cos\lambda_l z].$$

by means of $\sigma_r = 0$. The bending moment M_ϕ is computed from expression (3.77):

$$\left(\frac{M^{(2)}_\varphi}{M_0}\right)_{r=1}=\frac{h^2}{M_0}\int_{-1}^{1}\sigma^{(2)}_\varphi\,.\,z\,.\,dz=$$
$$=D(1+\nu)+6\nu\,.\,\beta^2\,\mathrm{Re}\sum_{l=1}^{\infty}b_l\,.\,\lambda_l\,.\,\frac{\sin^2\lambda_l}{\cos\lambda_l},\qquad(3.78)$$

which, after introduction of coefficients c_ℓ defined in expression (3.44), is converted into

$$\left(\frac{M^{(2)}_\varphi}{M_0}\right)_{r=1}=D(1+\nu)+\frac{3\nu}{2}\,.\,\beta^2\,\mathrm{Re}\sum_{l=1}^{\infty}c_l\,.\,\frac{tg^2\lambda_l}{\lambda^2_l}.\qquad(3.79)$$

Circumferential stress σ_ϕ on the boundary of the surface is determined from expression (3.77). It is found that

$$\left(\frac{\sigma^{(2)}_\varphi}{T}\right)_{\substack{r=1\\z=1}}=(1+\nu)\,(D+2\,\beta^2\,\mathrm{Re}\sum_{l=1}^{\infty}b_l\,.\,\lambda^2_l\sin\lambda_l),\qquad(3.80)$$

for which we may also write

$$\left(\frac{\sigma^{(2)}_\varphi}{T}\right)_{\substack{r=1\\z=1}}=(1+\nu)\left(D+{}^1\!/_2\,\beta^2\,\mathrm{Re}\sum_{l=1}^{\infty}c_l\frac{tg\,\lambda_l}{\lambda_l}\right).\qquad(3.81)$$

For the case of the very thin plate, expression (3.68) and (3.72) may /88 be used for expression (3.80) to write

$$\left(\frac{\sigma^{(2)}_\varphi}{T}\right)_{\substack{r=1\\z=1}}=(1+\nu)\Big\{-\frac{2}{3+\nu}+$$
$$+\frac{1}{\beta}\left(\frac{1488}{\pi^5}\zeta\,(5)\,\frac{2+\nu}{(3+\nu)^2}-\frac{112}{\pi^3}\zeta\,(3)\,\frac{1}{3+\nu}\right)+\ldots\Big\}.\qquad(3.82)$$

3.7 Reissner's Theory

While, in the classical theory of the bending of thin plates, only two variables -- the bending moment and the "reduced" transverse force -- may be prescribed on an edge (Refs. 32, 33), Reissner (Ref. 10) has undertaken to formulate a bending theory in which the three variables -- bending moment, torque, and transverse force -- may each separately assume a prescribed value on an edge. Green (Ref. 5) was the first to point out that from the three-dimensional theory we may derive a theory which essentially agrees with Reissner's theory. For this purpose, the coefficients a_k, b_ℓ, C and D are defined so that expressions (3.73), (3.74), and (3.75) each becomes zero. Since the three equations obtained in this way include threefold infinite unknowns, these equations may be fulfilled by assuming only a_o, C, and D to be unequal to zero. The following solution is found

$$\left.\begin{aligned}
a_0 &= \frac{-4/3}{(3+\nu)\beta^2 + \dfrac{16}{\pi^2}K\left(\dfrac{\pi\beta}{2}\right) + \dfrac{32}{\pi^2}},\\
C &= \frac{(1-\nu) - \dfrac{16}{\pi^2\beta^2}K\left(\dfrac{\pi\beta}{2}\right) + \dfrac{1}{\beta^2}\left(\dfrac{8-4\nu}{5} + \dfrac{8}{\pi^2} - 8\right)}{4(3+\nu) + \dfrac{64}{\pi^2\beta^2}K\left(\dfrac{\pi\beta}{2}\right) + \dfrac{128}{\pi^2\beta^2}},\\
D &= \frac{-2}{(3+\nu) + \dfrac{16}{\pi^2\beta^2}K\left(\dfrac{\pi\beta}{2}\right) + \dfrac{32}{\pi^2\beta^2}}.
\end{aligned}\right\} \qquad (3.83)$$

Setting b_ℓ equal to zero may be justified by taking the fact into considera- /89
tion that Reissner's theory is a thin-plate theory (β is large). As follows from the asymptotic expansion (3.49), the sums with the coefficients b_ℓ are then an order smaller in β than those with a_k in expression (3.73) and (3.75). Further, it is pointed out that in the expressions for the moments $M_r^{(2)}$, $M_{r\phi}^{(2)}$, and $M_\phi^{(2)}$ the sum over ℓ always includes the factor ν. For $\nu = 0$ these moments are thus *independent* of the stress state belonging to potentials B_1 and B_2. It may also be expected that for $\nu > 0$, the amount of this stressed state will be small. Disregard of a_1, a_2, ... with respect to a_o is justified by the form of expression (3.69).

If the third equation (3.83) is used to calculate the moment M_ϕ, we obtain the expression

$$\left(\frac{M_\varphi^{(2)}}{M_0}\right)_{r=1} = \frac{-2(1+\nu)}{3+\nu+\dfrac{16}{\pi^2\beta^2}K\left(\dfrac{\pi\beta}{2}\right)+\dfrac{32}{\pi^2\beta^2}}. \qquad (3.84)$$

This value agrees with the moment quoted by Reissner, if π is replaced by $\sqrt{10}$.

It should be remarked that the coefficients of expression (3.83) together with $a_k = 0$ and $b_\ell = 0$ for k, ℓ = 1, 2, ... make the expressions (3.73), (3.74) and (3.75) exactly zero for *every value of* β.

It is interesting to investigate how far this approximation theory of Reissner's, for large values of β, agrees with the asymptotic theory. For this purpose, the magnitude D from expression (3.83) given by D_{Reissner} is expanded with respect to $\frac{1}{\beta}$

$$D_{\text{Reissner}} = -\frac{2}{3+\nu} - \frac{16}{\pi}\,\frac{1}{(3+\nu)^2}\,\frac{1}{\beta} + \ldots, \tag{3.85}$$

from which by means of (3.68) it follows that

$$\frac{D^{(1)}_{\text{Reissner}}}{D^{(1)}_{\text{asympt}}} = \frac{\pi^4}{93\,\zeta(5)} \approx 1{,}01. \tag{3.86}$$

This agreement is very good. Deviations from Reissner's theory may therefore be expected only if /90

(a) β is not very great, i.e., for thick plates;
(b) the influence of the variables neglected by Reissner (a_1, a_2, ... and b_1, b_2, ...) becomes noticeable.

In thin plates, the values of a_1, a_2, ... will always be smaller than those of a_0, nor can the b_ℓ values always be neglected in thin plates. While, as already noted above, it can be expected that the effect of the b_ℓ values on the moments is negligible, this is never the case for the effect on the value of (σ_ϕ) r = z = 1.

Within the framework of the three-dimensional theory when b_ℓ and a_k are neglected, with k > 0, it is still more a question of restricting equations (3.37) to three equations with three unknowns a_0, C, and D, so that the following solution is obtained

$$\left.\begin{aligned}
a_0 &= \frac{-\dfrac{128}{\pi^4}}{(3+\nu)\beta^2 + \dfrac{16}{\pi^2}\,K\left(\dfrac{\pi\beta}{2}\right) + \dfrac{32}{\pi^2}},\\
C &= \frac{(1-\nu) - \dfrac{16}{\pi^2\beta^2}\,K\left(\dfrac{\pi\beta}{2}\right) - \dfrac{1}{\beta^2}\left(4\nu + \dfrac{32(1-\nu)}{\pi^2}\right)}{4(3+\nu) + \dfrac{64}{\pi^2\beta^2}\,K\left(\dfrac{\pi\beta}{2}\right) + \dfrac{128}{\pi^2\beta^2}},\\
D &= \frac{-2}{(3+\nu) + \dfrac{16}{\pi^2\beta^2}\,K\left(\dfrac{\pi\beta}{2}\right) + \dfrac{32}{\pi^2\beta^2}}.
\end{aligned}\right] \tag{3.87}$$

The stress distributions pertaining to expressions (3.83) and (3.87) are equivalent approximations for the thin plate. For the moment $M_\phi^{(2)}$, the same expression is found in both cases.

CHAPTER IV

NUMERICAL RESULTS FOR A PLATE LOADED IN BENDING /91

4.1 Introduction

The numerical calculations of the bending problem posed and formulated in Chapter III consist of the computation of the eigen values, the determination of coefficients (3.43), and the approximate solution of system (3.45). With a Poisson number of $\nu = {}^1/_4$, these computations are carried out for the values $\beta = 1, 2, 4$ and 6, which reduce system (3.45) to 10, 10, 8 and 6 real equations, respectively, with the same number of unknowns. Reference may be made to Chapter II for general considerations in regard to the calculations.

The results of these computations are collected in several tables. The eigen values and K*-values in accord with expression (2.7) are collected in Table VI. Tables VII give the coefficients $\chi_{\ell k}^{*(1)}$ and $\chi_{\ell k}^{*(2)}$ for different values of β, while the unknowns C, D, and c_ℓ are noted in Table VIII. The results in Table VIII are used to calculate the important mechanical quantities $\frac{\sigma_\phi}{T}$ from formulas (3.81) and (3.79) for r = z = 1, as well as $\frac{M_\phi}{M_0}$ for r = 1.

In figures 11 and 12, $\left(\frac{\sigma_\phi^{(2)}}{T}\right)_{r = z = 1}$ and $\left(\frac{M_\phi^{(2)}}{M_0}\right)_{r = 1}$ are given as functions of β. The superscript (2) represents the coefficient of cos 2_ϕ in the expression for the quantity. For purposes of comparison, these figures also show the values according to the first term of the asymptotic expansion and that according to Reissner, together with the classical values according to Goodier. The values according to Reissner are based on the solution (3.87).

4.2 Conclusions

The chief purpose of carrying out the computations in Chapters III and IV is to judge the accuracy which must be attributed to the theories of plate /98 bending which are used. Therefore, several quantities, which may well be considered the most characteristic, are compared with each other in the various theories, i.e., the value of $\sigma_\phi^{(2)}$ at the edge and $M_\phi^{(2)}$, the average value of $z\cdot\sigma_\phi^{(2)}$ over the thickness.

The graphs of Figures 11 and 12 give a true view of the reliability of the theories. It must be taken into consideration that the quantities noted here comprise only part of the corresponding mechanical variables. It is

TABLE VI

EIGEN VALUES AND K* VALUES

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$l = k+1$	$k'\pi$	λ_l	$K^*(k'\pi)$	$K^*(\lambda_l)$
1	1,5708	3,7488+i 1,3843	—0,4875	—0,7236—i 0,0753
2	4,7124	6,9500+i 1,6761	—0,7504	—0,8251—i 0,0354
3	7,8540	10,1192+i 1,8583	—0,8358	—0,8718—i 0,0209
4	10,9956	13,2773+i 1,9915	—0,8778	—0,8988—i 0,0138
5	14,1372	16,4299+i 2,0966	—0,9026	—0,9164—i 0,0099

$K^*(2\lambda_l)$	$K^*(4\lambda_l)$	$K^*(6\lambda_l)$
—0,8445—i 0,0489	—0,9173—i 0,0281	—0,9437—i 0,0197
—0,9057—i 0,0208	—0,9510—i 0,0113	—0,9669—i 0,0077
—0,9323—i 0,0117	—0,9652—i 0,0062	—0,9766—i 0,0042
—0,9472—i 0,0076	—0,9730—i 0,0040	—0,9819—i 0,0027
—0,9567—i 0,0065	—0,9779—i 0,0028	—0,9852—i 0,0019

TABLE VIIa

VALUES OF $\chi^{*(1)}_{\ell k}$ FOR $\beta = 1$

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k \ l	1	2	3	4
0	—0,0387+i 0,1484	—0,0161+i 0,0155	—0,0060+i 0,0038	—0,0028+i 0,0014
1	0,1058—i 0,1317	—0,0204+i 0,0660	—0,0121+i 0,0102	—0,0054+i 0,0030
2	0,0048—i 0,0139	0,0711—i 0,0691	—0,0124+i 0,0481	—0,0101+i 0,0085
3	0,0049—i 0,0040	—0,0024—i 0,0109	0,0577—i 0,0497	—0,0083+i 0,0402
4	0,0038—i 0,0020	0,0005—i 0,0025	—0,0040—i 0,0106	0,0502—i 0,0401

TABLE VIIb

VALUES OF $\chi^{*(2)}_{\ell k}$ FOR $\beta = 1$

k \ l	1	2	3	4
0	—0,0392+i 0,1666	—0,0172+i 0,0169	—0,0064+i 0,0041	—0,0030+i 0,0015
1	0,1621—i 0,3858	—0,0317+i 0,1369	—0,0219+i 0,0199	—0,0095+i 0,0056
2	—0,0489—i 0,0515	0,2224—i 0,3328	—0,0273+i 0,1615	—0,0290+i 0,0269
3	—0,0218—i 0,0151	—0,0609—i 0,0673	0,2715—i 0,3279	—0,0225+i 0,1911
4	—0,0120—i 0,0071	—0,0311—i 0,0185	—0,0683—i 0,0846	0,3157—i 0,3347

TABLE VIIc

VALUES OF $\chi_{\ell k}^{*(1)}$ FOR $\beta = 2$ /94

k \ l	1	2	3	4
0	—0,1121+i 0,3525	—0,0419+i 0,0388	—0,0157+i 0,0098	—0,0084+i 0,0036
1	0,3245—i 0,3664	—0,0688+i 0,2057	—0,0394+i 0,0325	—0,0176+i 0,0097
2	0,0164—i 0,0397	0,2408—i 0,2223	—0,0448+i 0,1637	—0,0573+i 0,0293
3	0,0159—i 0,0106	—0,0068—i 0,0368	0,2039—i 0,3699	—0,0307+i 0,1428
4	0,0124—i 0,0048	0,0022—i 0,0085	—0,0134—i 0,0375	0,1821—i 0,1422

TABLE VIId

VALUES OF $\chi_{\ell}^{*(2)}$ FOR $\beta = 2$

k \ l	1	2	3	4
0	—0,1372+i 0,5083	—0,0558+i 0,0535	—0,0207+i 0,0132	—0,0097+i 0,0048
1	0,8553—i 2,1594	—0,1799+i 0,7668	—0,1241+i 0,1118	—0,0540+i 0,0317
2	—0,3393—i 0,3244	1,4608—i 2,1931	—0,1818+i 1,0657	—0,1924+i 0,1775
3	—0,1602—i 0,0965	—0,4263—i 0,4672	1,8948—i 2,2852	—0,1590+i 1,3351
4	—0,0911—i 0,0455	—0,2247—i 0,1321	—0,4916—i 0,6092	2,2795—i 2,4116

TABLE VIIe

VALUES OF $\chi_{\ell k}^{*(1)}$ FOR $\beta = 4$ /95

k \ l	1	2	3	4
0	—0,3648+i 1,0436	—0,1301+i 0,1182	—0,0492+i 0,0303	—0,0233+i 0,0112
1	1,1487—i 1,2013	—0,2529+i 0,7244	—0,1422+i 0,1181	—0,0640+i 0,0350
2	0,0664—i 0,1354	0,8876—i 0,7919	—0,1699+i 0,6037	—0,1325+i 0,1086
3	0,0600—i 0,0352	—0,0210—i 0,1354	0,7670—i 0,6272	—0,1187+i 0,5378
4	0,0462—i 0,0153	—0,0099—i 0,0314	—0,0480—i 0,1412	0,6936—i 0,5347

TABLE VIIf

VALUES OF $\chi_{\ell}^{*(2)}$ FOR $\beta = 4$

k \ l	1	2	3	4
0	—0,6688+i 2,3792	—0,2665+i 0,2531	—0,0993+i 0,0628	—0,0465+i 0,0228
1	5,7893—i 14,6314	—1,2387+i 0,5226	—0,8506+i 1,3926	—0,3702+i 0,2168
2	—2,4538—i 2,3313	10,6096—i 15,8675	—1,3320+i 7,7413	—1,4020+i 1,2909
3	—1,1979—i 0,7136	—3,1856—i 3,5016	14,2437—i 17,1326	—1,2054+i 10,0380
4	—0,6905—i 0,3400	—1,7034—i 1,0048	—3,7334—i 4,6392	17,3889—i 18,3614

TABLE VIIg

VALUES OF $\chi^{*(1)}_{\ell k}$ FOR $\beta = 6$ /96

k \ l	1	2
0	—0,7625+i 2,1144	—0,2682+i 0,2222
1	2,4872—i 2,5229	—0,5531+i 1,5624
2	0,1517—i 0,2899	1,9442—i 1,7125

TABLE VIIh

VALUES OF $\chi^{*(2)}_{\ell k}$ FOR $\beta = 6$

k \ l	1	2
0	—1,9244+i 6,7725	—0,7628+i 0,7228
1	18,6323—i 46,8700	—3,9989+i 16,7959
2	—8,0104—i 7,6309	34,8230—i 51,9744

TABLE VIII

VALUES OF C, D, AND c_ℓ /97

β	C	D	c_1
1	0,2197	—0,9887	1,6699—i 0,4180
2	0,1589	—0,8281	0,2556—i 0,0748
4	0,1133	—0,7271	0,0369—i 0,0124
6	0,0959	—0,6905	0,0107—i 0,0037

c_2	c_3	c_4
0,7474+i 0,0052	0,3347+i 0,1136	0,1065+i 0,1086
0,0923—i 0,0001	0,1110—i 0,0155	—0,0152+i 0,0623
0,0105+i 0,0024	0,0026+i 0,0026	—
0,0028+i 0,0004	—	—

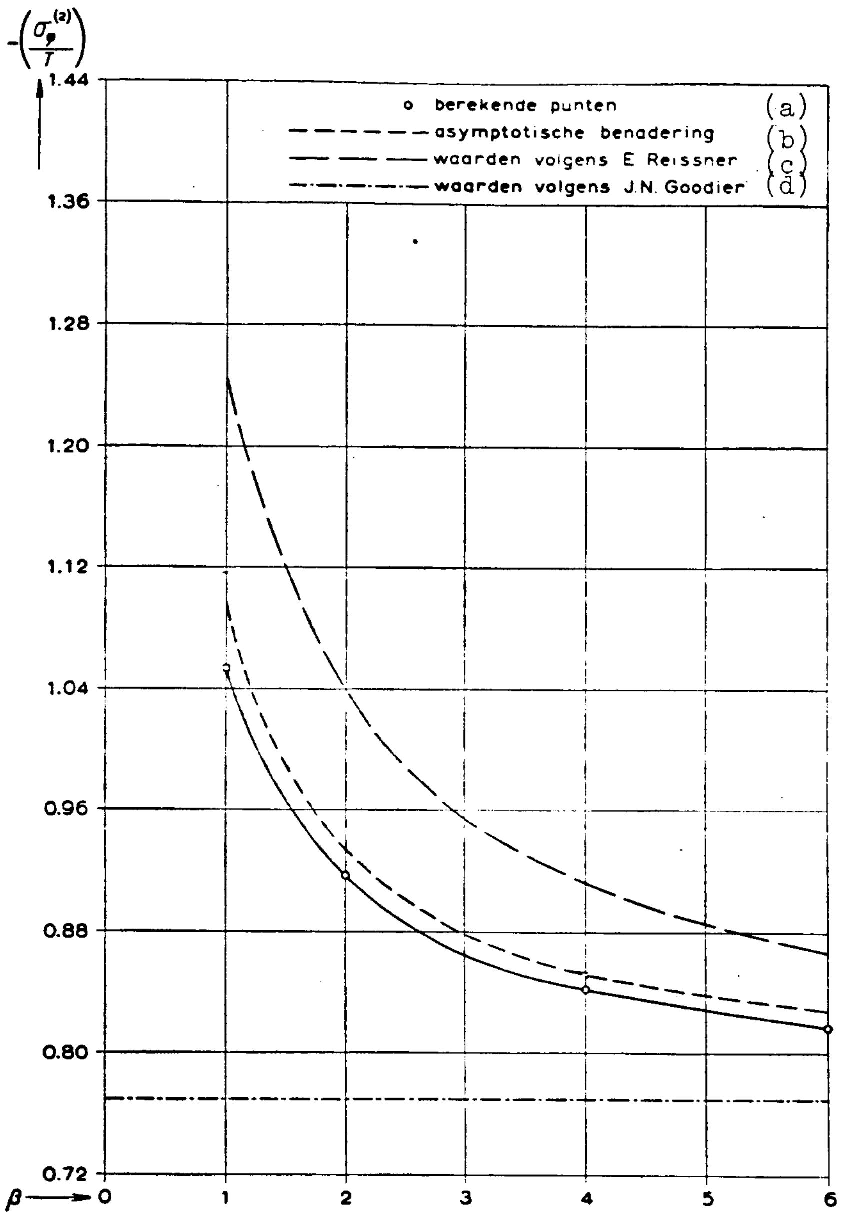

Figure 11

Value of $\sigma_\phi^{(2)}$ Along the Edge of the Hole on the External Plane /99

(a) - Calculated points; (b) - Asymptotic approximation; (c) - Values according to E. Reissner; (d) - Values according to J. N. Goodier.

always true that

$$\sigma_\varphi = \sigma_\varphi^{(0)} + \sigma_\varphi^{(2)} \cos 2\varphi,$$
$$M_\varphi = M_\varphi^{(0)} + M_\varphi^{(2)} \cos 2\varphi$$

and in the various theories $\sigma_\phi^{(0)}$ and $M_\phi^{(0)}$ have the same value. The relative

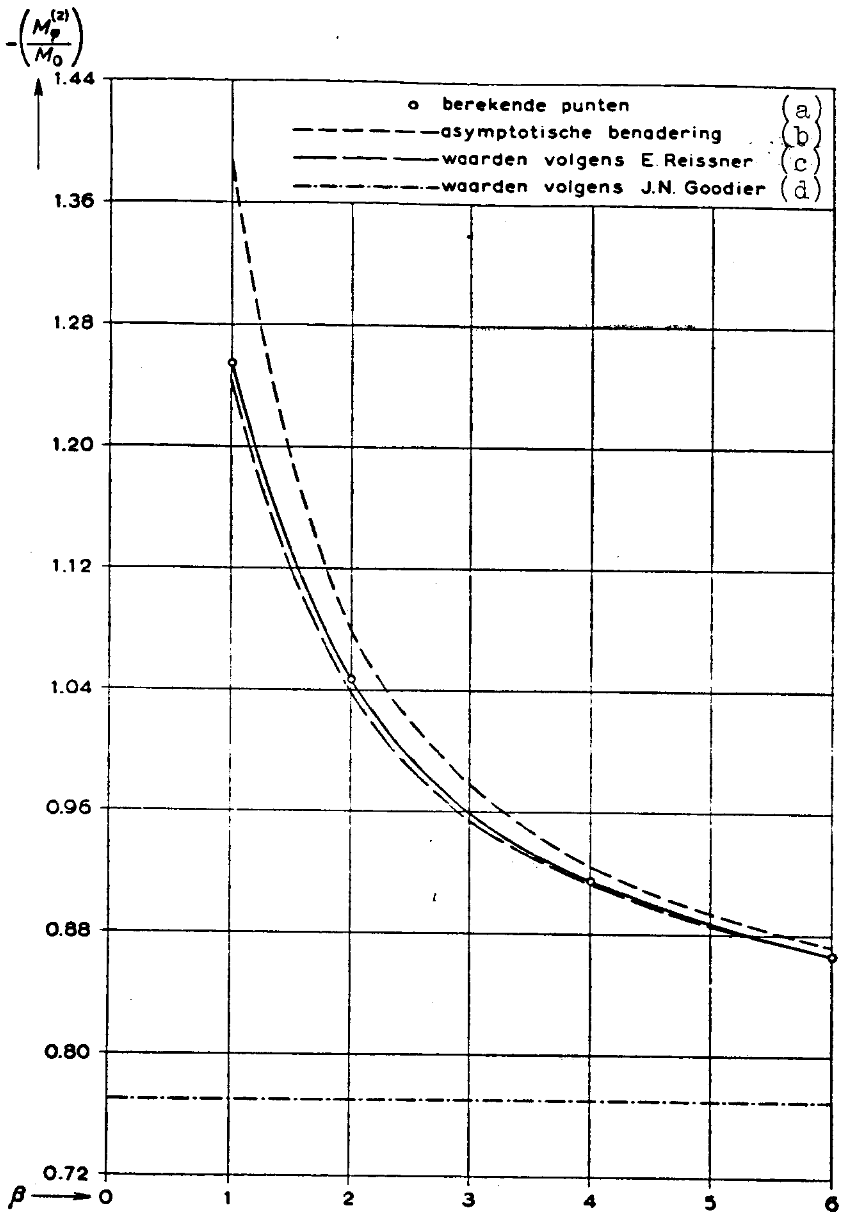

Figure 12

Moment $M_\phi^{(2)}$ Along the Edge of the Hole /100

(a) - Calculated points; (b) - Asymptotic approximation; (c) - Values according to E. Reissner; (d) - Values according to J. N. Goodier.

deviations in the maxima from the mechanical variables will thus always be smaller than the deviations from the noted variables. But these differences have an effect on the conclusions which can be drawn:

(a) the classic solution has only a very restricted application;

(b) the theory of Reissner is excellent as an average-value theory;
(c) on the surface, only the three-dimensional theory is reliable.

Although these conclusions are based on the results which are obtained in calculating a special problem, the bending of a large plate with a circular hole, it may nevertheless be expected that their scope is much wider and that they are of general application in plate bending problems.

Reissner's theory is adequate to compute the moments which occur, as is clear from Figure 12. The maximum error which appears in the moments is no more than 2%. The fact that methods of calculation according to Reissner for *thin* plates give good results may be explained by means of formula (3.86). It is not automatically clear, however, that this theory also gives the approximately true value for the moments in plates *of medium thickness*. If the passage to the limit $\beta \to 0$ (which in itself is meaningless) is made in formula (3.84), we obtain the expression

$$\left(\frac{M_{\varphi}^{(2)}}{M_0}\right)_{r=0} \to -2, \tag{4.1}$$

the exact value for the thick plate. This result, together with the value obtained from the numerical calculations when $\beta = 1$, gives an explanation of the good agreement found over the whole region of the ratio of hole-diameter to plate-thickness. /101

It goes without saying that for typical stress concentrations in thick plates, like $\left(\frac{\sigma_{\phi}}{T}\right)_{z = r = 1}$, every average-value theory must be insufficient. This is also true of Reissner's theory. It is remarkable, however, that here the error in the value found with the classical theory is just as great as that found by Reissner's theory. In a plate which is not too thin the error here is on the order of magnitude of 7%.

CHAPTER V

THE MATHAR-SOETE METHOD /102

5.1 Introduction

This chapter will discuss the theoretical foundation of an experimental method of determining the residual stresses on the external surface of a structure by measuring the changes in strain which occur after a cylindrical hole is bored. This method is known in the literature as the Mathar-Soete method and, in short, reduces to gluing three small strain gauges on the face of the structure; they are symmetrical to, and at the same distance from, the hole which is to be bored. From the indications of the strain gauges, it is possible to compute the principal stresses on the surface of the structure, as well as the angle which the principal stresses make with a fixed system of axes.

It is outside of the scope of this paper to enter into the details of the experimental execution of this method. Conclusive calculation of the stresses from the requisite formulas likewise remains untreated. For these things the existing literature (especially Refs. 12 and 16) is to be consulted.

The problem that is to be treated in this chapter, however, is the effect of the three-dimensional state of stress on the indications of the strain gauges. In the literature, to be sure, it is unusual to calculate from the standpoint of a two-dimensional state of stress, although certain publications have been conscious of the fact that deviations from this state of stress should be taken into consideration (Refs. 15, 6).

If in the two-dimensional state of tension the principal stresses are given by T_1 and T_2, and the T_1 direction coincides with the X-axis, then according to expression (1.24) it is true for the whole plate that /103

$$\left.\begin{aligned} \sigma_r &= \frac{T_1+T_2}{2}+\frac{T_1-T_2}{2}\cos 2\varphi, \\ \sigma_\varphi &= \frac{T_1+T_2}{2}-\frac{T_1-T_2}{2}\cos 2\varphi, \\ \tau_{r\varphi} &= -\frac{T_1-T_2}{2}\sin 2\varphi. \end{aligned}\right] \qquad (5.1)$$

After the hole has been bored, the two-dimensional stresses are (compare expressions [1.33] and [1.34a])

$$\left.\begin{aligned} \sigma_r &= \frac{T_1+T_2}{2}\left(1-\frac{a^2}{r^2}\right)+\frac{T_1-T_2}{2}\cos 2\varphi\left\{1+3\frac{a^4}{r^4}-4\frac{a^2}{r^2}\right\}, \\ \sigma_\varphi &= \frac{T_1+T_2}{2}\left(1+\frac{a^2}{r^2}\right)+\frac{T_1-T_2}{2}\cos 2\varphi\left\{-1-3\frac{a^4}{r^4}\right\}, \\ \tau_{r\varphi} &= \frac{T_1-T_2}{2}\sin 2\varphi\left\{-1+3\frac{a^4}{r^4}-2\frac{a^2}{r^2}\right\}. \end{aligned}\right] \qquad (5.2)$$

The stress changes to be measured are thus

$$\left.\begin{aligned} \sigma_r' &= \frac{T_1+T_2}{2}\left(-\frac{a^2}{r^2}\right)+\frac{T_1-T_2}{2}\cos 2\varphi\left\{-4\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right\}, \\ \sigma_\varphi' &= \frac{T_1+T_2}{2}\left(\frac{a^2}{r^2}\right)+\frac{T_1-T_2}{2}\cos 2\varphi\left\{-3\frac{a^4}{r^4}\right\}, \\ \tau_{r\varphi}' &= \frac{T_1-T_2}{2}\sin 2\varphi\left\{-2\frac{a^2}{r^2}+3\frac{a^4}{r^4}\right\}. \end{aligned}\right] \qquad (5.3)$$

The prime (') indicates that these are *changes* in the variables. From expression (5.3) the tension changes may be computed, and from them the predicted indications of the strain gauges. The thus found formulas are, however, not entirely complete for the simplified assumption of a two-dimensional state of stress, and will now be made complete. /104

To simplify the notation, the primes on the stress and strain changes will be omitted in this chapter and T_2 will be set equal to zero. Furthermore, the radius of the hole is introduced as the unit of length.

5.2 General Formulas

5.2.1 Transformation of the System of Axes of the Strain Gauge

Stress T acts in the X-direction. Rectangular strain gauge 2d in width and ℓ in length is glued at the angle α (see Figure 13). A new (x, y) system whose x-axis makes an angle α with the X-axis gives the rectangular coordinates of points on the strain gauge. Since the strain must be averaged over the gauge, it is necessary to define the magnitudes of deformation with respect to this system. It is clear that

$$\left.\begin{aligned} x &= r\cos(\varphi-\alpha), \\ y &= r\sin(\varphi-\alpha), \end{aligned}\right] \tag{5.4}$$

whence

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$$\begin{aligned} dx &= \cos(\varphi-\alpha)\ dr - r\sin(\varphi-\alpha)\ d\varphi, \\ dy &= \sin(\varphi-\alpha)\ dr + r\cos(\varphi-\alpha)\ d\varphi. \end{aligned}$$

For the tensions in the new coordinates

$$\left.\begin{aligned} \varepsilon_x &= \varepsilon_r\cos^2(\varphi-\alpha) + \varepsilon_\varphi\sin^2(\varphi-\alpha) - \gamma_{\varphi r}\sin(\varphi-\alpha)\cos(\varphi-\alpha), \\ \varepsilon_y &= \varepsilon_r\sin^2(\varphi-\alpha) + \varepsilon_\varphi\cos^2(\varphi-\alpha) + \gamma_{\varphi r}\sin(\varphi-\alpha)\cos(\varphi-\alpha), \end{aligned}\right] \tag{5.5}$$

holds true; from this it directly follows that

$$\varepsilon_x + \varepsilon_y = \varepsilon_r + \varepsilon_\varphi. \tag{5.6}$$

5.2.2 Relative Change in Resistance

In a homogeneous strain field ε_x, ε_y the relative change in strain gauge resistance, $\frac{\Delta R}{R}$, consists of two components, parallel to the longitudinal and transverse strain, respectively

$$\frac{\Delta R}{R} = K_l\,\varepsilon_x + K_d\,\varepsilon_y.$$

K_ℓ and K_d are defined by this formula.

In an inhomogeneous elastic field, integration must be carried out over the whole filament of the strain gauge. The relative change in resistance now becomes

$$\frac{\Delta R}{R} = \frac{K}{L}\int_0^L \varepsilon\, ds, \tag{5.7}$$

in which K is the strain gauge constant for a single filament, L the whole length of the filament, and ε is the strain in the filament direction.

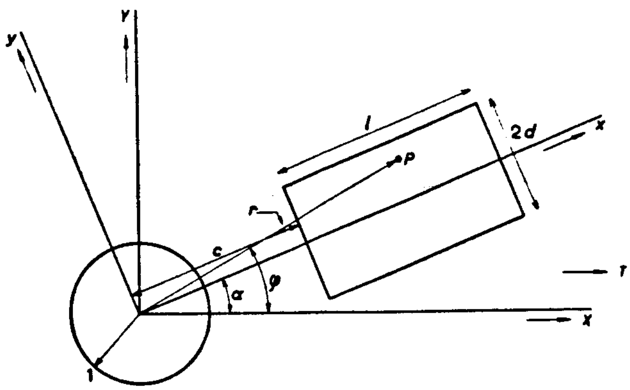

Figure 13

Location of Strain Gauge with Respect to the Hole

If the strain gauge consists of an adequately large number of windings, expression (5.7) may, with good approximation, be replaced by /106

$$\frac{\Delta R}{R} = K_l \bar{\bar{\varepsilon}}_x + {}^1/_2 K_d (\bar{\varepsilon}_y)_{x=c} + {}^1/_2 K_d (\bar{\varepsilon}_y)_{x=c+l}, \tag{5.8}$$

in which the first term represents the (most important) contribution in the longitudinal direction of the gauge, while the second and third terms are derived from the transverse filament elements located at the ends of the band. The quantities $\bar{\bar{\varepsilon}}_x$ and $\bar{\varepsilon}_y$ are defined by

$$\bar{\bar{\varepsilon}}_x = \frac{1}{2dl} \int_{-d}^{+d} \int_{c}^{c+l} \varepsilon_x \, dx \, dy, \tag{5.9}$$

$$\bar{\varepsilon}_y = \frac{1}{2d} \int_{-d}^{+d} \varepsilon_y \, dy, \tag{5.10}$$

where c is the distance from the front edge of the strain gauge to the origin (see Figure 13).

If the measuring bridge is as usual set at the standard constant 2, the value measured is $\rho = {}^1/_2 \frac{\Delta R}{R}$.

Since ε_x and ε_y are parallel to $\frac{T}{E}$, we can write

$$\rho = {}^1/_2 \frac{\Delta R}{R} = P \cdot \frac{T}{E} + (Q + R) \cdot \frac{T}{E} \cos 2\alpha. \tag{5.11}$$

It is the aim of the calculations to find expressions for P, Q, and R. The (artificial) division of $\frac{T}{E}$ cos 2α into (Q + R) indicates the difference in nature between the elementary and the non-elementary part of the coefficient.

5.2.3 Some Integrals

Before proceeding to the actual calculation, we will here give several elementary integrals which play an important role in computing the variables.

These integrals are defined by means of the following relationships:

$$\left.\begin{aligned}
\cos 2(\varphi-\alpha) &= \cos^2(\varphi-\alpha)-\sin^2(\varphi-\alpha) = \frac{x^2-y^2}{x^2+y^2},\\
\cos(4\varphi-2\alpha) &= \cos 2\alpha \left\{ 2\left(\frac{x^2-y^2}{x^2+y^2}\right)^2 - 1 \right\} + \\
&\quad + \sin 2\alpha \left(\frac{x^2-y^2}{x^2+y^2}\right)^2 \cdot \frac{4\,xy}{x^2-y^2},\\
\cos 2\varphi &= \cos 2(\varphi-\alpha)\ \cos 2\alpha - \sin 2(\varphi-\alpha)\ \sin 2\alpha,\\
\sin 2\varphi &= \sin 2(\varphi-\alpha)\ \sin 2\alpha + \sin 2(\varphi-\alpha)\ \cos 2\alpha.
\end{aligned}\right] \qquad (5.12)$$

It is found that

$$\int_{-d}^{+d} dy \,.\, \frac{\cos 2(\varphi-\alpha)}{r^2} = -\frac{2d}{x^2+d^2}, \qquad (5.13)$$

$$\int_{-d}^{+d} dy \int_{c}^{c+l} dx \,.\, \frac{\cos 2(\varphi-\alpha)}{r^2} = 2\left(\operatorname{arctg}\frac{d}{c} - \operatorname{arctg}\frac{d}{c+l}\right), \qquad (5.14)$$

$$\int_{-d}^{+d} dy \,.\, \frac{\cos(2\alpha-4\varphi)}{r^2} = \cos 2\alpha \,.\, \frac{2d\,(x^2-d^2)}{(x^2+d^2)^2}, \qquad (5.15)$$

$$\left.\begin{aligned}
\int_{-d}^{+d} dy \int_{c}^{c+l} dx\ \frac{\cos(2\alpha-4\varphi)}{r^2} &= \cos 2\alpha \,.\, 2d \left[\frac{c}{c^2+d^2} - \right.\\
&\quad \left. - \frac{c+l}{d^2+(c+l)^2}\right],
\end{aligned}\right] \qquad (5.16)$$

$$\int_{-d}^{+d} dy\ \frac{\cos(2\alpha-4\varphi)}{r^4} = \cos 2\alpha \,.\, {}^2/_3 \,.\, \frac{d(3x^2-d^2)}{(x^2+d^2)^3}, \qquad (5.17)$$

$$\left.\begin{aligned}
\int_{-d}^{+d} dy \int_{c}^{c+l} dx\ \frac{\cos(2\alpha-4\varphi)}{r^4} &= \cos 2\alpha \,.\, \frac{2d}{3}\left[\frac{c}{(c^2+d^2)^2} - \right.\\
&\quad \left. - \frac{c+l}{(d^2+(c+l)^2)^2}\right].
\end{aligned}\right] \qquad (5.18)$$

5.3 Calculating Deformations

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5.3.1 Pure Tension Loading

The stress variations along the surface z = 1 are determined by the following formulas (see also expressions [1.104], [1.105])

$$\frac{\sigma_r}{T} = -\frac{1}{2}\frac{1}{r^2} + \cos 2\varphi \Bigg[6C \cdot \frac{1}{r^4} - D \left\{ \frac{4(1+\nu)}{r^2} + \frac{12\nu}{\beta^2}\frac{1}{r^4} \right\} - \frac{2}{r^2}\sum_k a_k(-1)^k \left\{ \frac{3K_2(k\pi\beta r)}{K_2(\lambda\pi\beta)} - (k\pi\beta r)\frac{K_1(k\pi\beta r)}{K_2(k\pi\beta)} \right\} + \mathrm{Re}\sum_l b_l \left\{ 2(1-\nu)\cos\lambda_l \left(\frac{1}{r^2}\frac{6K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)} - \frac{\lambda_l\beta}{r}\frac{K_1(\lambda_l\beta r)}{K_2(\lambda_l\beta)} \right) + 2\cos\lambda_l \cdot \lambda_l^2\beta^2 \frac{K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)} \right\} \Bigg], \tag{5.19}$$

$$\frac{\sigma_\varphi}{T} = \frac{1}{2}\frac{1}{r^2} + \cos 2\varphi \Bigg[-6C\frac{1}{r^4} + D\left\{ \frac{12\nu}{\beta^2}\frac{1}{r^4} \right\} + \frac{2}{r^2}\sum_k a_k(-1)^k \left\{ \frac{3K_2(k\pi\beta r)}{K_2(k\pi\beta)} - (k\pi\beta r)\frac{K_1(k\pi\beta r)}{K_2(k\pi\beta)} \right\} + \mathrm{Re}\sum_l b_l \left\{ -2(1-\nu)\cos\lambda_l \left(\frac{1}{r^2}\frac{6K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)} - \frac{\lambda_l\beta}{r}\frac{K_1(\lambda_l\beta r)}{K_2(\lambda_l\beta)} \right) + 2\nu\cos\lambda_l \cdot \lambda_l^2\beta^2\frac{K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)} \right\} \Bigg], \tag{5.20}$$

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$$\frac{\tau_{r\varphi}}{T} = \sin 2\varphi \Bigg[6C\frac{1}{r^4} - D\left\{ \frac{2(1+\nu)}{r^2} + \frac{12\nu}{\beta^2}\frac{1}{r^4} \right\} + \frac{1}{r^2}\sum_k a_k(-1)^k \left\{ -\frac{6K_2(k\pi\beta r)}{K_2(k\pi\beta)} + (k\pi\beta r)\frac{K_1(k\pi\beta r)}{K_2(k\pi\beta)} - \frac{k^2\pi^2\beta^2 r^2}{2}\frac{K_2(k\pi\beta r)}{K_2(k\pi\beta)} \right\} + \mathrm{Re}\sum_l b_l \cdot \left\{ -4(1-\nu)\cos\lambda_l \left(-\frac{3}{r^2}\frac{K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)} + \frac{\lambda_l\beta}{r}\frac{K_1(\lambda_l\beta r)}{K_2(\lambda_l\beta)} \right) \right\} \Bigg]. \tag{5.21}$$

From formulas (5.12) to (5.21), ε_r, ε_ϕ, and $\gamma_{r\phi}$ are calculated by means of Hooke's law

$$\left.\begin{aligned}
\varepsilon_r &= \frac{\sigma_r - \nu\sigma_\varphi}{E},\\
\varepsilon_\varphi &= \frac{\sigma_\varphi - \nu\sigma_r}{E},\\
\gamma_{r\varphi} &= \frac{\tau_{r\varphi}}{G} = \frac{\tau_{r\varphi}}{E} 2(1+\nu).
\end{aligned}\right] \tag{5.22}$$

The first two formulas hold true here because $\sigma_z = 0$ over the surface. For *extra* deformations it is now found that /110

$$\left.\begin{aligned}
\frac{E\varepsilon_r}{T} = &-{}^1\!/_2 \frac{(1+\nu)}{r^2} + \cos 2\varphi \Bigg[6(1+\nu) C \frac{1}{r^4} -\\
&- D\,.\,4(1+\nu) \left\{ \frac{1}{r^2} + \frac{3\nu}{\beta^2} \frac{1}{r^4} \right\} -\\
&\frac{2(1+\nu)}{r^2} \sum_k a_k (-1)^k \left\{ \frac{3K_2(k\pi\beta r)}{K_2(k\pi\beta)} - (k\pi\beta r) \frac{K_1(k\pi\beta r)}{K_2(k\pi\beta)} \right\} +\\
&+ 2(1-\nu^2)\, \mathrm{Re} \sum_l b_l \cos\lambda_l \,.\, \left\{ \frac{1}{r^2} \frac{6K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)} -\right.\\
&\left. - \frac{\lambda_l\beta}{r} \frac{K_1(\lambda_l\beta r)}{K_2(\lambda_l\beta)} \right\}\\
&+ 2(1-\nu^2)\, \mathrm{Re} \sum_l b_l \cos\lambda_l \,.\, \lambda_l^2\beta^2 \frac{K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)} \Bigg],
\end{aligned}\right] \tag{5.23}$$

$$\left.\begin{aligned}
\frac{E\varepsilon_\varphi}{T} = &{}^1\!/_2 \frac{(1+\nu)}{r^2} + \cos 2\varphi \Bigg[-6(1+\nu) C \frac{1}{r^4} +\\
&+ D\,.\,4\nu(1+\nu) \left\{ \frac{1}{r^2} + \frac{3}{\beta^2} \frac{1}{r^4} \right\} +\\
&+ \frac{2(1+\nu)}{r^2} \sum_k a_k (-1)^k \left\{ \frac{3K_2(k\pi\beta r)}{K_2(k\pi\beta)} -\right.\\
&\left. - (k\pi\beta r) \frac{K_1(k\pi\beta r)}{K_2(k\pi\beta)} \right\} - 2(1-\nu^2)\, \mathrm{Re} \sum_l b_l \cos\lambda_l \,.\\
&\left\{ \frac{1}{r^2} \frac{6K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)} - \frac{\lambda_l\beta}{r} \frac{K_1(\lambda_l\beta r)}{K_2(\lambda_l\beta)} \right\} \Bigg],
\end{aligned}\right] \tag{5.24}$$

$$\frac{E\gamma_{r\varphi}}{T} = \sin 2\varphi \Bigg[12(1+\nu)C\frac{1}{r^4} - D\Big\{ 4(1+\nu)^2\frac{1}{r^2} + \frac{24\nu(1+\nu)}{\beta^2}\frac{1}{r^4}\Big\} + \frac{2(1+\nu)}{r^2}\sum_k a_k(-1)^k.$$

$$\Big\{ -6\frac{K_2(k\pi\beta r)}{K_2(k\pi\beta)} + (k\pi\beta r)\frac{K_1(k\pi\beta r)}{K_2(k\pi\beta)} - \tag{5.25}$$

$$- \frac{k^2\pi^2\beta^2 r^2}{2}\cdot\frac{K_2(k\pi\beta r)}{K_2(k\pi\beta)}\Big\} - 8(1-\nu^2)\,\mathrm{Re}\sum_l b_l.$$

$$\Big[\cos\lambda_l \Big\{\frac{\lambda_l\beta}{r}\frac{K_1(\lambda_l\beta r)}{K_2(\lambda_l\beta)} - \frac{3}{r^2}\frac{K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)}\Big\}\Big]\Bigg].$$

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From strains (5.23) to (5.25), ε_x and ε_y are calculated by means of expressions (5.5) and (5.6). After some calculation it is found that the parts of ε_x and ε_y *symmetrical* with respect to the y-axis are

$$\frac{E\varepsilon_{xs}}{T} = -(1+\nu)\Bigg[\tfrac{1}{2}\frac{\cos 2(\varphi-\alpha)}{r^2} - \Big(6C - \frac{12\nu}{\beta^2}D\Big)\frac{\cos(4\varphi-2\alpha)}{r^4} + 2(1+\nu)D.\frac{\cos(4\varphi-2\alpha)}{r^2} +$$

$$+ 2(1-\nu)D.\frac{\cos 2\varphi}{r^2}\Bigg] + (1+\nu)\sum_k a_k(-1)^k.$$

$$\Big\{ -\frac{6K_2(k\pi\beta r)}{K_2(k\pi\beta)}\frac{\cos(4\varphi-2\alpha)}{r^2} + \frac{k\pi\beta}{2}\frac{K_1(k\pi\beta r)}{K_2(k\pi\beta)}\Big(\frac{\cos 2\alpha}{r} + \tag{5.26}$$

$$+ \frac{3\cos(4\varphi-2\alpha)}{r}\Big) + \frac{k^2\pi^2\beta^2}{4}\frac{K_2(k\pi\beta r)}{K_2(k\pi\beta)}(\cos 2\alpha -$$

$$-\cos(4\varphi-2\alpha)\Big\} + (1-\nu^2)\,\mathrm{Re}\sum_l b_l\cos\lambda_l\Big\{\frac{12K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)}.$$

$$.\frac{\cos(4\varphi-2\alpha)}{r^2} + \lambda_l\beta\frac{K_1(\lambda_l\beta r)}{K_2(\lambda_l\beta)}\Big(\frac{\cos 2\alpha}{r} -$$

$$-\frac{3\cos(4\varphi-2\alpha)}{r}\Big) + 2\lambda_l^2\beta^2\frac{K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)}\cos 2\varphi\cos^2(\varphi-\alpha)\Big\},$$

and /112

$$\frac{E\varepsilon_{ys}}{T} = (1+\nu)\Bigg[\tfrac{1}{2}\frac{\cos 2(\varphi-\alpha)}{r^2} - \Big(6C - \frac{12\nu}{\beta^2}D\Big)\frac{\cos(4\varphi-2\alpha)}{r^4} + 2(1+\nu)D\frac{\cos(4\varphi-2\alpha)}{r^2} -$$

$$-2(1-\nu)D\frac{\cos 2\varphi}{r^2}\Bigg] - (1+\nu)\sum_k a_k(-1)^k \tag{5.27}$$

$$\begin{aligned}
&\Big\{-\frac{6K_2(k\pi\beta r)}{K_2(k\pi\beta)}\,\frac{\cos(4\varphi-2\alpha)}{r^2}+\frac{k\pi\beta}{2}\,\frac{K_1(k\pi\beta r)}{K_2(k\pi\beta)}\left(\frac{\cos 2\alpha}{r}+\right.\\
&\left.+\frac{3\cos(4\varphi-2\alpha)}{r}\right)+\frac{k^2\pi^2\beta^2}{4}\,\frac{K_2(k\pi\beta r)}{K_2(k\pi\beta)}(\cos 2\alpha-\\
&-\cos(4\varphi-2\alpha))\Big\}+(1-\nu^2)\,\mathrm{Re}\sum_l b_l\cos\lambda_l\\
&\Big\{-\frac{12\,K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)}\,\frac{\cos(4\varphi-2\alpha)}{r^2}-\\
&-\lambda_l\beta\,\frac{K_1(\lambda_l\beta r)}{K_2(\lambda_l\beta)}\left(\frac{\cos 2\alpha}{r}-\frac{3\cos(4\varphi-2\alpha)}{r}\right)+\\
&+2\lambda_l^2\beta^2\,\frac{K_2(\lambda_l\beta r)}{K_2(\lambda_l\beta)}\,\cos 2\varphi\,.\,\sin^2(\varphi-\alpha)\Big\}.
\end{aligned} \tag{5.27}$$

From expressions (5.26) and (5.27) $\bar{\bar{\varepsilon}}_x$ and $\bar{\varepsilon}_y$ are calculated in accord with expressions (5.10) and (5.11). Here use is made of the integrals which are computed in Section 5.23. Then ρ, in which the coefficients P, Q, and R occur, is determined by means of expressions (5.9) and (5.12). For convenience these coefficients are broken up as follows:

$$\begin{aligned}
P &= P_1+P_{21}+P_{22},\\
Q &= Q_1+Q_{21}+Q_{22}+Q_3+Q_{41}+\\
&\qquad +Q_{42}+Q_5+Q_{61}+Q_{62},\\
R &= R_1+R_{21}+R_{22}+R_3+R_{41}+R_{42},
\end{aligned} \tag{5.28}$$

in which the various portions are given by /113

$$\begin{aligned}
P_1 &= -\frac{K_l(1+\nu)}{4dl}\Big\{\mathrm{arctg}\frac{d}{c}-\mathrm{arctg}\frac{d}{c+l}\Big\},\\
P_{21} &= \frac{K_d(1+\nu)}{8}\cdot\frac{1}{c^2+d^2},\\
P_{22} &= \frac{K_d(1+\nu)}{8}\cdot\frac{1}{(c+l)^2+d^2},
\end{aligned} \tag{5.29}$$

$$\begin{aligned}
Q_1 &= \frac{K_l(1+\nu)}{l}\Big\{\frac{c}{(c^2+d^2)^2}-\\
&\qquad -\frac{c+l}{((c+l)^2+d^2)^2}\Big\}\left(C-\frac{2\nu}{\beta^2}D\right),\\
Q_{21} &= -\frac{K_d(1+\nu)}{2}\cdot\frac{3c^2-d^2}{(c^2+d^2)^3}\left(C-\frac{2\nu}{\beta^2}D\right),\\
Q_{22} &= -\frac{K_d(1+\nu)}{2}\cdot\frac{3(c+l)^2-d^2}{((c+l)^2+d^2)^3}\left(C-\frac{2\nu}{\beta^2}D\right);
\end{aligned} \tag{5.30}$$

$$\left.\begin{aligned}
Q_3 &= -\frac{K_l(1+\nu)^2}{l}\left(\frac{c}{c^2+d^2}-\frac{c+l}{(c+l)^2+d^2}\right).D,\\
Q_{41} &= \frac{K_d(1+\nu)^2}{2}\cdot\frac{c^2-d^2}{(c^2+d^2)^2}.D,\\
Q_{42} &= \frac{K_d(1+\nu)^2}{2}\cdot\frac{(c+l)^2-d^2}{((c+l)^2+d^2)^2}.D;
\end{aligned}\right]\tag{5.31}$$

$$\left.\begin{aligned}
Q_5 &= -\frac{K_l(1-\nu^2)}{dl}\left(\operatorname{arctg}\frac{d}{c}-\operatorname{arctg}\frac{d}{c+l}\right).D,\\
Q_{61} &= -\frac{K_d(1-\nu^2)}{2}\cdot\frac{1}{c^2+d^2}.D,\\
Q_{62} &= -\frac{K_d(1-\nu^2)}{2}\cdot\frac{1}{(c+l)^2+d^2}.D;
\end{aligned}\right]\tag{5.32}$$

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$$\left.\begin{aligned}
R_1 &= \frac{K_l(1+\nu)}{2d.l}\sum_k\frac{a_k(-1)^k}{K_2(k\pi\beta)}\left[\int_{-d}^{+d}dy\int_c^{c+l}dx\right.\\
&\left\{-\frac{3K_2(k\pi\beta r)}{r^2}\left(1-\frac{8x^2y^2}{r^4}\right)+\frac{k\pi\beta}{r}K_1(k\pi\beta r)\left(1-\right.\right.\\
&\left.\left.\left.-\frac{6x^2y^2}{r^4}\right)+k^2\pi^2\beta^2K_2(k\pi\beta r)\frac{x^2y^2}{r^4}\right\}\right],\\
R_{21} &= -\frac{K_d(1+\nu)}{4d}\sum_k\frac{a_k(-1)^k}{K_2(k\pi\beta)}\left[\int_{-d}^{+d}dy\right.\\
&\left\{-\frac{3K_2(k\pi\beta r)}{r^2}\left(1-\frac{8x^2y^2}{r^4}\right)+\frac{k\pi\beta}{r}K_1(k\pi\beta r)\left(1-\right.\right.\\
&\left.\left.\left.-\frac{6x^2y^2}{r^4}\right)+k^2\pi^2\beta^2K_2(k\pi\beta r).\frac{x^2y^2}{r^4}\right\}_{x=c}\right],\\
R_{22} &= -\frac{K_d(1+\nu)}{4d}\sum_k\frac{a_k(-1)^k}{K_2(k\pi\beta)}\left[\int_{-d}^{+d}dy\,\{---\}_{x=c+l}\right],
\end{aligned}\right]\tag{5.33}$$

$$\left.\begin{aligned}
R_3 &= \frac{K_l(1-\nu^2)}{2dl}\operatorname{Re}\sum_l\frac{b_l\cos\lambda_l}{K_2(\lambda_l\beta)}\left[\int_{-d}^{+d}dy\int_c^{c+l}dx\right.\\
&\left\{\frac{6K_2(\lambda_l\beta r)}{r^2}\left(1-\frac{8x^2y^2}{r^4}\right)-\frac{\lambda_l\beta}{r}K_1(\lambda_l\beta r)\left(1-\right.\right.\\
&\left.\left.\left.-\frac{12x^2y^2}{r^4}\right)+\lambda_l^2\beta^2K_2(\lambda_l\beta r)\left(1-\frac{y^2}{r^2}-\frac{2x^2y^2}{r^4}\right)\right\}\right],\\
R_{41} &= -\frac{K_d(1-\nu^2)}{4d}\operatorname{Re}\sum_l\frac{b_l\cos\lambda_l}{K_2(\lambda_l\beta)}\left[\int_{-d}^{+d}dy\right.
\end{aligned}\right]\tag{5.34}$$

$$\left\{ \frac{6K_2(\lambda_l\beta r)}{r^2}\left(1-\frac{8x^2y^2}{r^4}\right)-\frac{\lambda_l\beta}{r}K_1(\lambda_l\beta r)\left(1-\right.\right.$$

$$\left.\left.-\frac{12x^2y^2}{r^4}\right)-\lambda^2{}_l\beta^2 K_2(\lambda_l\beta r)\cdot\frac{y^2}{r^2}\left(1-\frac{2y^2}{r^2}\right)\right\}_{x=c}\Bigg],$$

$$R_{42}=-\frac{K_d(1-\nu^2)}{4d}\ \mathrm{Re}\sum_l\frac{b_l\cos\lambda_l}{K_2(\lambda_l\beta)}\left[\int_{-d}^{+d}dy\{---\}_{x=c+l}\right]. \tag{5.34}$$

If very narrow strain gauges are used we may in the first approximation assume $d = K_d = 0$. In this case, the forms (5.33) to (5.34) are considerably simplified since the integrals may be considered definite. /115

The limiting case $d = 0$ gives

$$P_1=-\frac{K_l(1+\nu)}{4}\cdot\frac{1}{c(c+l)},$$

$$P_{21}=P_{22}=0;$$

$$Q_1=K_l(1+\nu)\left(\frac{3c^2+3cl+l^2}{c^3(c+l)^3}\right)\left(C-\frac{2\nu}{\beta^2}D\right),$$

$$Q_3=-K_l(1+\nu)^2\cdot\frac{1}{c(c+l)}\cdot D,$$

$$Q_5=-K_l(1-\nu^2)\cdot\frac{1}{c(c+l)}\cdot D;$$

$$Q_{21}=Q_{22}=Q_{41}=Q_{42}=Q_{61}=Q_{62}=0;$$

$$R_1=-\frac{K_l(1+\nu)}{l}\sum_k a_k(-1)^k\left\{\frac{K_2(k\pi\beta c)}{K_2(k\pi\beta)}\ \frac{1}{c}-\right.$$

$$\left.-\frac{K_2(k\pi\beta(c+l))}{K_2(k\pi\beta)}\ \frac{1}{c+l}\right\},$$

$$R_{21}=R_{22}=0,$$

$$R_3=\frac{K_l(1-\nu^2)}{l}\ \mathrm{Re}\sum_l b_l\cos\lambda_l$$

$$\left\{\frac{\lambda_l\beta\{K_1(\lambda_l\beta(c+l))-K_1(\lambda_l\beta c)\}}{K_2(\lambda_l\beta)}-\right.$$

$$\left.-\left(\frac{2}{c+l}\ \frac{K_2(\lambda_l\beta(c+l))}{K_2(\lambda_l\beta)}-\frac{2}{c}\ \frac{K_2(\lambda_l\beta c)}{K_2(\lambda_l\beta)}\right)\right\},$$

$$R_{41}=R_{42}=0. \tag{5.35}$$

It is clear that constant P is entirely defined by the two-dimensional theory (see expression [5.29]). The three-dimensional perturbation thus is expressed only in the constant Q + R. /116

5.3.2 Some Numerical Results for Pure Tension

The voluminous numerical computation was carried out for some cases. This can give adequate insight, on the one hand, into the three-dimensional perturbation of the two-dimensional theory and, on the other hand, into the effect of the finite width of the strain gauge. Here ν is always taken to equal $^1/_4$, while $c = {}^5/_3$ and $\ell = 4$.

The comparison is based on strain gauges of different widths, all of which have the same *resultant* strain gauge constant K_r in a *state of linear stress*. The symbol χ is introduced for the ratio of transverse sensitivity K_d to longitudinal sensitivity K_ℓ (see Section 5.22). The value of 2.01 found by van Dongen is used for the strain gauge constant K_r. For a length/width ratio of 2, the experimental value of 0.028 is established for the transverse sensitivity χ of a strain gauge. For a narrow strain gauge, χ is assumed to be zero. Table IX gives the computational results for the four combinations $d = 0$ or 1 and $\beta = {}^1/_2$ or ∞.

TABLE IX

		Q	R	Q+R	P
d = 0	$\beta = \infty$	–0,1798	0	–0,1798	–0,0665
	$\beta = {}^1/_2$	–0,2449	0,0615	–0,1834	–0,0665
d = 1	$\beta = \infty$	–0,1388	0	–0,1388	–0,0552
	$\beta = {}^1/_2$	–0,1701	0,0270	–0,1431	–0,0552

The table indicates that the effect of plate thickness on coefficient Q + R is slight. Although perhaps a greater effect can appear for other relationships of the measurements, in particular if the strain gauge is brought still closer to the edge and is shorter, it may be expected with certainty that the correction to the result of the two-dimensional theory is always of subordinate importance. Of much greater significance is the effect of the finite width of the strain gauge, as may be gathered from the results which are given for the same value of β and with various values of d. /117

5.3.3 Pure Bending

We treat the bending of a performated plate loaded by a bending moment M_0 operating along an edge parallel to the Y axis. The formulas of

Chapter III can be applied here.

There is no point in repeating the calculations, which are quite similar to those in Section 5.3.1. Merely the results of the computations are given. We find

$$\left.\begin{aligned}
P_1 &= -\frac{K_l(1+\nu)}{4dl}\left\{\operatorname{arctg}\frac{d}{c} - \operatorname{arctg}\frac{d}{c+l}\right\}\\
P_{21} &= \frac{K_d(1+\nu)}{8}\,\frac{1}{c^2+d^2},\\
P_{22} &= \frac{K_d(1+\nu)}{8}\,\frac{1}{(c+l)^2+d^2},
\end{aligned}\right]\tag{5.36}$$

$$\left.\begin{aligned}
Q_1 &= \frac{K_l(1+\nu)}{l}\left\{\frac{c}{(c^2+d^2)^2} - \right.\\
&\quad \left. - \frac{c+l}{((c+l)^2+d^2)^2}\right\}\left\{-C + \frac{4+\nu}{6\beta^2}D\right\},\\
Q_{21} &= -\frac{K_d(1+\nu)}{2}\cdot\frac{3c^2-d^2}{(c^2+d^2)^3}\left\{-C+\frac{4+\nu}{6\beta^2}D\right\},\\
Q_{22} &= -\frac{K_d(1+\nu)}{2}\,\frac{3(c+l)^2-d^2}{((c+l)^2+d^2)^3}\\
&\qquad \left\{-C+\frac{4+\nu}{6\beta^2}D\right\},
\end{aligned}\right]\tag{5.37}$$

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$$\left.\begin{aligned}
Q_3 &= -\frac{K_l(1-\nu^2)}{4l}\left(\frac{c}{c^2+d^2} - \frac{c+l}{(c+l)^2+d^2}\right).D,\\
Q_{41} &= \frac{K_d(1-\nu^2)}{8}\cdot\frac{c^2-d^2}{(c^2+d^2)^2}.D,\\
Q_{42} &= \frac{K_d(1-\nu^2)}{8}\,\frac{(c+l)^2-d^2}{((c+l)^2+d^2)^2}.D;
\end{aligned}\right]\tag{5.38}$$

$$\left.\begin{aligned}
Q_5 &= \frac{K_l(1-\nu^2)}{4dl}\left\{\operatorname{arctg}\frac{d}{c} - \operatorname{arctg}\frac{d}{c+l}\right\}.D,\\
Q_{61} &= \frac{K_d(1-\nu^2)}{8}\cdot\frac{1}{c^2+d^2}.D,\\
Q_{62} &= \frac{K_d(1-\nu^2)}{8}\cdot\frac{1}{(c+l)^2+d^2}.D;
\end{aligned}\right]\tag{5.39}$$

$$
\left.
\begin{aligned}
R_1 &= \frac{K_l(1+\nu)}{2dl}\sum_k \frac{a_k(-1)^k}{K_2(k'\pi\beta)}\left[\int_{-d}^{+d} dy \int_c^{c+l} dx\right. \\
&\left\{-\frac{3K_2(k'\pi\beta r)}{r^2}\left(1-\frac{8x^2y^2}{r^4}\right)+\frac{k'\pi\beta}{r}K_1(k'\pi\beta r)\left(1-\right.\right. \\
&\qquad\left.\left.\left.-\frac{6x^2y^2}{r^4}\right)+k'^2\pi^2\beta^2 K_2(k'\pi\beta r)\frac{x^2y^2}{r^4}\right\}\right], \\
R_{21} &= -\frac{K_d(1+\nu)}{4d}\sum_k \frac{a_k(-1)^k}{K_2(k'\pi\beta)}\left[\int_{-d}^{+d} dy\right. \\
&\left\{-\frac{3K_2(k'\pi\beta r)}{r^2}\left(1-\frac{8x^2y^2}{r^4}\right)+\frac{k'\pi\beta}{r}K_1(k'\pi\beta r)\left(1-\right.\right. \\
&\qquad\left.\left.\left.-\frac{6x^2y^2}{r^4}\right)+k'^2\pi^2\beta^2 K_1(k'\pi\beta r)\frac{x^2y^2}{r^4}\right\}_{x=l}\right], \\
R_{22} &= -\frac{K_d(1+\nu)}{4d}\sum_k \frac{a_k(-1)^k}{K_2(k'\pi\beta)} \\
&\qquad\left[\int_{-d}^{+d} dy \{---\}_{x=c+l}\right];
\end{aligned}
\right]
\qquad (5.40)
$$

$$
\left.
\begin{aligned}
R_3 &= \frac{K_l(1-\nu^2)}{2dl}\,\mathrm{Re}\sum_l \frac{b_l\sin\lambda_l}{K_2(\lambda_l\beta)}\left[\int_{-d}^{+d} dy \int_c^{c+l} dx\right. \\
&\left\{\frac{6K_2(\lambda_l\beta r)}{r^2}\left(1-\frac{8x^2y^2}{r^4}\right)-\frac{\lambda_l\beta}{r}K_1(\lambda_l\beta r)\left(1-\right.\right. \\
&\left.\left.\left.-\frac{12x^2y^2}{r^4}\right)-\lambda_l^2\beta^2 K_2(\lambda_l\beta r)\left(1-\frac{y^2}{r^2}-\frac{2x^2y^2}{r^4}\right)\right\}\right], \\
R_{41} &= -\frac{K_d(1-\nu^2)}{4d}\,\mathrm{Re}\sum_l \frac{b_l\sin\lambda_l}{K_2(\lambda_l\beta)}\left[\int_{-d}^{+d} dy\right. \\
&\left\{\frac{6K_2(\lambda_l\beta r)}{r^2}\left(1-\frac{8x^2y^2}{r^4}\right)-\frac{\lambda_l\beta}{r}K_1(\lambda_l\beta r)\left(1-\right.\right. \\
&\left.\left.\left.-\frac{12x^2y^2}{r^4}\right)-\lambda^2{}_l\beta^2K_2(\lambda_l\beta r)\cdot\frac{y^2}{r^2}\left(1-\frac{2y^2}{r^2}\right)\right\}_{x=l}\right], \\
R_{42} &= -\frac{K_d(1-\nu^2)}{4d}\,\mathrm{Re}\sum_l \frac{b_l\sin\lambda_l}{K_2(\lambda_l\beta)} \\
&\qquad\left[\int_{-d}^{+d} dy \{---\}_{x=c+l}\right].
\end{aligned}
\right]
\qquad (5.41)
$$

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5.3.4. Adaptation of Reissner's Theory

The results of section 5.3.2 demonstrate that with pure tension the plate thickness has merely slight effect on the measured specific variation in resistance of the strain gauge. Based on the results reported in Chapter IV, however, it is to be expected that with pure bending the effect of plate thickness will be much greater, as is also ascertained by the experiments (Ref. 16). Finally, to obtain qualitative insight into the connection existing between specific resistance variation and plate thickness, without again having to carry out voluminous computations, the factor Q + R is calculated by means of the Reissner theory for a linear strain gauge; factor P is already explained entirely by the classical theory. Our starting point in this is the variable (3.87), while the given strain gauge parameters are

$$d = 0, \ c = 2, \ \ell = 4, \ K_r = 2.01.$$

Figure 14 represents the result. For factor Q + R the deviation from the classical theory is very important.

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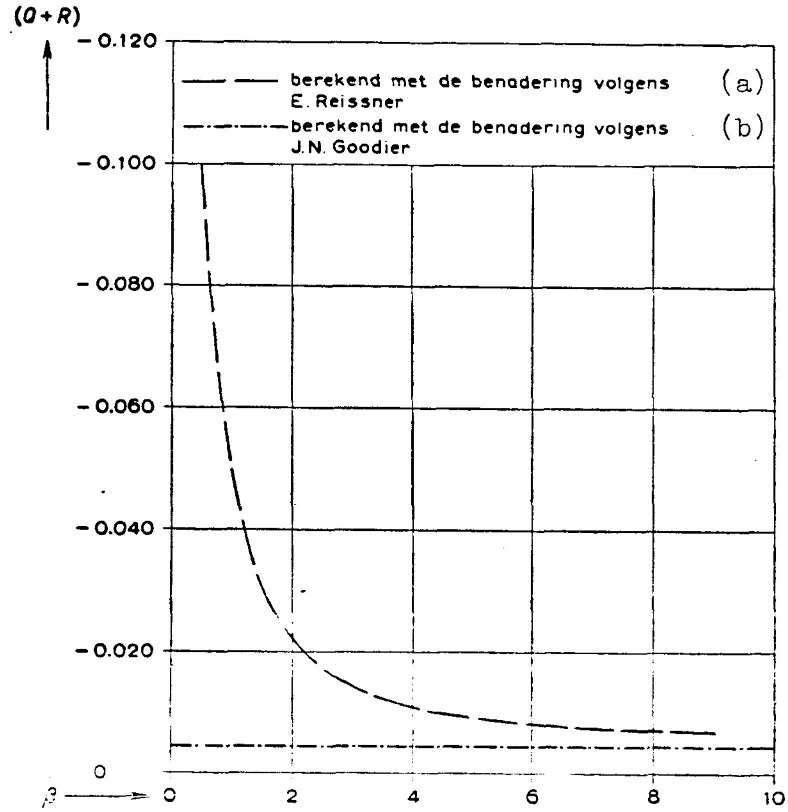

Figure 14

Coefficient (Q + R) for a Linear Strain Gauge in the Case of Pure Bending. (For given strain gauge parameters see this page.)

(a) - Calculated with approximation according to E. Reissner; (b) - Calculated with approximation according to J. N. Goodier

From the results in Figure 11, Reissner's theory seems rather unsuited for giving insight into displacements on the surface. The great differences which may exist on the hole-edge $r = 1$ will, however, at values of $r > \sim 2$ already be substantially reduced because the differences with respect to the classical theory decrease exponentially with radius r according to expression (3.26). This phenomenon is clearly illustrated by Figure 3 for the case of pure tension.

5.4 Comparison of Theoretical and Experimental Values /121

It is interesting to find out whether the complicated theory which is developed here is capable of explaining the known experimental quantities. A group of measuremental results, for which all parameters are accurately known, is the one given by van Dongen (Ref. 15). Van Dongen has restricted his measurements to strain measurements on a plate with the following strain gauge characteristics

$$\nu = 0{,}274;\ c = {}^5/_3,\ l = 4,\ d = 1;\ K_r = 2{,}01,\ \chi = 0{,}028$$

He finds

$$P = -0{,}0590,$$
$$Q + R = -0{,}1394.$$

The theoretical values computed in § 5.3.2 are related to

$$\nu = 0{,}25;\ c = {}^5/_3,\ l = 4,\ \ d = 1; K_r = 2{,}01,\ \chi = 0{,}028$$

and are (see Table IX)

$$P = -0{,}0552,$$
$$Q + R = -0{,}1431.$$

It may be said that the agreement is very satisfactory.

The deviations may have their origin in the following causes:

1. ν does not have the same value;
2. tolerances in the experiment must be adjusted to the values of ℓ and c;
3. the theoretical values are surface averages, but must be linear integrals along the strain gauge filament (see § 5.2.2;
4. the manner of loading in this experiment is not ideal in nature, as is assumed in theory.

In summary we may say that relative resistance variations in the strain gauge are always predicted with good approximation by three-dimensional theory. In most cases, however, we will be able to make do with an even simpler theory. Thus, according to § 5.3.2, for pure tension the classic two-dimensional theory is almost always sufficiently accurate, while for pure bending the correc- /122 tion according to Reissner's theory is for the most part adequate (§ 5.3.4). The numerical parameters given in Chapters II and IV are certainly sufficient in any concrete case to ascertain to what degree the simplified theory may be reliably applied.

REFERENCES

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1. Kirsch, G. Die Theorie der Elastizität und die Bedürfnisse der Festigkeitslehre (Theory of Elasticity and the Requirements of Stress Analysis). Z.V.D.I., 42, 797, 1898.
2. Filon, L. N. G. On an Approximative Solution for the Bending of a Beam of Rectangular Cross-section. Phil. Trans. Roy. Sec. A 201, 63, 1903.
3. Love, A. E. H. Mathematical Theory of Elasticity, 4th Ed., Cambridge, 1934.
4. Green, A. E. Three-dimensional Stress Systems in Isotropic Plates I. Phil. Trans. Roy. Soc. Lond. A, 240, 561, 1949.
5. Green, A. E. The Elastic Equilibrium of Isotropic Plates and Cylinders. Proc. Roy. Soc. A, 195, 533, 1949.
6. Sternberg, E., Sadowsky, M. A. Three-dimensional Solution for the Stress Concentration Around a Circular Hole in a Plate of Arbitrary Thickness. J. Appl. Mech., 16, 27, 1949.
7. Bickley, W. G. The Effect of a Hole in a Bent Plate. Phil. Mag., 6, 48, 1014, 1924.
8. Goodier, J. N. The Influence of Circular and Elliptical Holes on the Transverse Flexure of Elastic Plates. Phil. Mag., 7, 22, 69, 1936.
9. Reissner, E. On the Calculation of Three-dimensional Corrections for the Two-dimensional Theory of Plane Stress. Proc. Fifteenth Semi-ann. East. Photoel. Confer., 23, 1942.
10. Reissner, E. The Effect of Transverse-Shear Deformation on the Bending of Elastic Plates. J. Appl. Mech., 12, 69, 1945.
11. Mathar, J. Ermittlungen von Eigenspannungen durch Messung von Bohrlochverformungen (Residual Stress Data from Measurement of Borehole Deformation). Arch. f. d. Eisenhüttenw., 6, 277, 1932.
12. Soete, W., Vancrombrugge, R. An Industrial Method for the Determination of Residual Stresses. Proc. Soc. Exp. Str. An., 8, 17, 1950.
13. Soete, W. Measurement and Relaxation of Residual Stresses. Weld. J. Res. Suppl., 354 s, 1949.
14. Soete, W., Vancrombrugge, R. Het meten van eigenspanningen in diepte (Measurement of Residual Stresses in Depth). Lastijdschrift, 4, 17, 1948.
15. v. Dongen, L. Bepalingen van las- en gietspanningen (Determining Welding and Casting Stresses). Ingenieur, 63, 0.45, 1951.
16. Boiten, R. G., ten Cate, W. A Routine Method for the Measurement of Residual Stresses in Plates. Appl. Sc. Res., A, 3, 318, 1953.
17. Biezeno, C. B., Grammel, R. Technische Dynamik (Engineering Dynamics), I, § 17, Berlin, 1939. /124
18. Boussinesq, J. Applications des Potentials (Potential Applications), Paris, 1885.
19. Watson, G. N. Theory of Besselfunctions, 2nd Ed., Cambridge, 1944.
20. Whittaker, E. T., Watson, G. N. Modern Analysis, 4th Ed., Cambridge, 1946.
21. Hütte, I, 25th Ed., 168, Berlin, 1925.
22. Papkovich, P. F. De uitdrukking voor de oplossing van de fundamentele elasticiteits-vergelijkingen in harmonische functies (in R) [The Expression for the Solution of the Basic Elasticity Equations in Harmonic Functions (in R)]. Akad. Nauk. SSSR, Phys. M.S., 10, 1425, 1932.

23. Eubanks, R. A., Sternberg, E. On the Completeness of the Boussinesq-Papkovich Stress Functions. J. Rat. Mech. Anal., 5, 735, 1956.
24. Table of the Besselfunctions $J_0(z)$ and $J_1(z)$ for Complex Arguments, Ed. by Comp. Lab., N.B.S., New York, 1947.
25. Table of the Besselfunctions $Y_0(z)$ and $Y_1(z)$ for Complex Arguments, Ed. by Comp. Lab., N.B.S., New York, 1950.
26. Hayashi, K. Fünfstellige Tafeln der Kreis und Hyperbelfunktionen (Five-Place Tables of Circular and Hyperbolic Functions). Berlin, 1931.
27. Horvay, G. The End Problem of Rectangular Strips. J. Appl. Mech., 20, 87, 1953.
28. Hillmann, A. P., Salzer, H. E. Roots of sin z = z. Phil. Mag., 7, 34, 575, 1943.
29. Fadle, J. Die Selbstspannungs-Eigenwertfunktionen der Quadratischen Scheibe (Residual Stress Eigen-Value Functions of Square Plates). Ing. Arch. II, 125, 1941.
30. Kellogg, O. D. Foundations of Potential Theory, Chapt. VI, New York, 1953.
31. Knopp, K. Theorie und Anwendungen der unendlichen Reihen (Theory and Application of Infinite Series), 3rd Ed., § 50, Berlin, 1931.
32. Love, A. E. H. Math. Theory of Elasticity, 4th Ed., Chapt. XXII, Cambridge, 1934.
33. Biezeno, C. B., Grammel, R. Technische Dynamik (Engineering Dynamics)., Chapt. VI, Berlin, 1939.
34. Dougall, J. An Analytical Theory of the Equilibrium of an Isotropic Elastic Rod of Circular Section. Trans. Roy. Soc. Edinb., 49, 895, 1913.

SUMMARY

In the classical theory of elasticity, the solution of problems concerning stretching and bending of plates is usually based on the theory of generalized plane stress. It is to be expected that in some cases the deviations from the classical theory, due to the three-dimensional character of the stress distribution will become important. For the special cases of stretching and bending of an infinite plate with a circular cylindrical hole, a rigorous analysis is developed in this thesis. The method of investigation is based on the use of complex eigenfunctions, whose general theory was developed some years ago by Green (Ref. 5). It is the purpose of this investigation to examine the effect of the plate-thickness on the stress distribution and to discuss the errors due to the approximate character of current two-dimensional theories. Although this investigation covers only some special problems and is not intended to give a general theory, it is believed that the conclusions arrived at here will hold more generally. Apart from this fundamental aspect of the present investigation, the calculations made for the three-dimensional stress distribution also provide a theoretical basis for the method of Mathar-Soete for the measurement of residual stresses.

In Chapter I the analytical theory is given for the stretched plate. The solution is found by means of a superposition of plane stress solutions and three-dimensional solutions. The boundary conditions at the hole lead to an infinite system of equations for the coefficients of superposition. The solution of this system is approximated by truncation to a finite system. In the case of a very thin plate, the three-dimensional solution becomes a perturbation of the plane stress solution. It is then possible to give the asymptotic development for the coefficients.

The component solutions used are derived from three harmonic stress-functions. In the appendix to Chapter I a proof is given for the completeness of this system of stress functions for a class of regions including the one under consideration.

In Chapter II the numerical results for the case of stretching are given corresponding to a value of 0.25 for Poisson's ratio.

The more important results are shown in the figures 2 to 9. In most of these figures the results are compared with those obtained by Green for the case $\beta = 1$, where β is the ratio of the diameter of the hole to the plate thickness. In general the two computations show very good agreement. In Figure 2 the mean value $\bar{\sigma}_z$ of the normal stress acting perpendicular to the plate faces is given as a function of β. The value of $\bar{\sigma}_z$, given in this figure can be considered as a measure of the deviation from the plane-stress and the plane-strain state. The dependence of σ_z, for $\beta = 1$ on the radial distance is shown in Figure 3. From this graph some conclusions can be drawn which are of interest for the interpretation of the results obtained with the Mathar-Soete-method. In Figure 4 the stress σ_z itself is given as a function of the thickness coordinate z for several values of β. It can be seen that at $\beta = 1/2$ the value of σ_z at

the middle plane agrees approximately with the value of the plane strain solution. The maximum value of σ_z, which occurs at the middle plane is shown in Figure 5 as a function of $1/\beta$. Figures 6 to 9 refer to the tangential stress σ_ϕ at the hole. Figure 6 gives the mean value, Figure 7 the value at the faces of the plate, and Figure 8 the value at the middle plane, all as a function of β. In Figure 9 the distribution over the thickness is shown. Of particular interest are the results of Figure 7 from which it can be seen that the stress concentration at the boundary differs by more than 10% from that given by classical theory.

In Chapter III the analytical theory of bending of plates is given. Special attention is paid to the case of the thin plate, and the relation of the present results to those based on the theory of Reissner is discussed extensively.

The numerical results for the case of bending are given in Chapter IV. In Figure 11 the most interesting stress, the tangential stress σ_ϕ at the edge of the hole, is shown as a function of β. Comparisons are made with the corresponding results obtained by Goodier and by Reissner. Figure 12 displays the β-dependence of the tangential moment M_ϕ. It is apparent from the comparison with the theory of Reissner, that this theory gives a very good approximation for the stress-resultants, but by no means for the stresses themselves.

Chapter V deals with application of the general theory presented here to the Mathar-Soete method of evaluation of residual stresses. The effect of the plate thickness on the evaluation of the strain gauge readings is discussed in some detail. The main results are as follows:

(a) for the case of stretching, the effect of the thickness is not very important in practical instances;
(b) for the case of bending, there is a strong effect of the thickness, as can be seen from Figure 14, where the thickness-dependent part of the resistance deviation has been shown for a very narrow strain-gauge in the first approximation;
(c) the effect of the width of the strain-gauge is very important so that a surface integration over the strain-gauge is necessary.

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